3. Quantum filtering theory $\Rightarrow$ Stochastic master equation (SME)

Setting: $\cdot\ \langle e(0), \cdot\, e(0)\rangle = \varphi_R$ on $\mathcal{B}(\Gamma_s(L^2(\mathbb{R}^+)))$
$\cdot\ \mathrm{Tr}(\rho\,\cdot) = \varphi_s$ on $\mathcal{B}(\mathcal{H}_s)$, $\dim(\mathcal{H}_s) = N < \infty$ $\Big\} \Rightarrow (\mathcal{A}, \varphi)$

$\varphi = \varphi_s \otimes \varphi_R$.

Input field $\xrightarrow{}$ $A_t, A_t^\dagger$ | $\boxed{\text{System (H.P.)}\quad \hat{j}_t(X) = U_t^*(X\otimes I)U_t}$ | Output field $U_t^* A_t U_t$, $U_t^* A_t^\dagger U_t$ $\to$ $\boxed{H.D}$ | Observation $Y_t = U_t^* Z_t U_t$, $(A_t + A_t^\dagger)$ $\to$ $\boxed{\text{Filter}}$ $\xrightarrow{\text{optimal estimation of } j_t(X)}$ $\pi_t(X) = \varphi(j_t(X) \mid \Upsilon_t)$, $vN\{Y_s \mid 0\le s\le t\}$.

H-P eq. $dU_t = \cdots$

Properties:

① Self-nondemolition: Y_t can be measured in a single realization.

②. nondemolition: $j_t(X)$ and Y_t is well defined.
$\Rightarrow \pi_t(X)$ is well defined.

③. $\varphi(j_t(X)) = \varphi(\pi_t(X))$.

$\boxed{\text{Objective}}$: Obtain an explicit expression of $\pi_t(X)$ in terms of $\{Y_s\}_{s\in[0,t]}$.

Lemma: $\forall X \in \mathcal{B}(\mathcal{H}_s)\otimes \mathbb{I}$, define $\underbrace{\omega_t(X) = \varphi(U_t^* X U_t)}_{\text{H-P eq.}}$, then:

$$\pi_t(X) = \varphi(j_t(X)\mid \Upsilon_t) = U_t^*\,\omega_t(X\mid \mathcal{Z}_t)\,U_t.$$

$\mathcal{Z}_t = vN\{Z_s \mid 0\le s\le t\}$ $\quad \Upsilon_t = vN\{Y_s \mid 0\le s\le t\}$. $\Upsilon_t = U_t^* \mathcal{Z}_t U_t$.

Proof: $\forall S \in \mathcal{Z}_t$, $U_t^* S U_t \in \Upsilon_t$.

$$\varphi(U_t^*\,\omega_t(X\mid\mathcal{Z}_t)\,\overbrace{U_t\, U_t^*}^{I} S U_t) = \omega_t(\omega_t(X\mid \mathcal{Z}_t) S)$$

$$\stackrel{\text{def}}{=} \omega_t(XS) = \varphi(\underbrace{U_t^* X U_t}_{j_t(X)} U_t^* S U_t) \stackrel{\text{def}}{=} \varphi(\varphi(j_t(X)\mid \Upsilon_t)\,U_t^* S U_t)\ \#$$

$$\omega_t(X) = \varphi(U_t^* X U_t)$$

Try: Apply B.F. for $\omega_t(X|Z_t) = \dfrac{\varphi(U_t^* X U_t | Z_t)}{\varphi(U_t^* U_t | Z_t)}$

$U_t \in Z_t'$!!!

$\Rightarrow Y_t = U_t^* Z_t U_t = Z_t \Rightarrow$ no interaction sys + field.

$\Rightarrow$ no info.

$\Rightarrow$ We cannot use U_t as change-of-state operator.

$\boxed{\text{Task:}}$ Find $V_t \in M_n(\mathcal{N}(Z_t))$ s.t. $\varphi(U_t^* X U_t) = \varphi(V_t^* \hat{} X \hat{} V_t)$

Recall: $\mathcal{N}(Z_t)$: all normal operators affiliated to Z_t commutative.

$M_N(\mathcal{N}(Z_t))$: $\begin{bmatrix} V_{11} & \cdots & V_{1N} \\ \vdots & \ddots & \vdots \\ V_{N1} & \cdots & V_{NN} \end{bmatrix}$ $V_{ij} \in \mathcal{N}(Z_t)$.

Lemma: $dV_t = D_t dA_t^\dagger + F_t dA_t + G_t dt$,

$d\tilde{V}_t = \tilde{D}_t dA_t^\dagger + \tilde{F}_t dA_t + \tilde{G}_t dt$, $V_0 = \tilde{V}_0$, $X_t = X_t \otimes \mathbb{1}$

Then: $V_t\, v \otimes e(0) = \tilde{V}_t\, v \otimes e(0)$, $\forall v \in \mathcal{H}_S$, $\forall t \in [0,T]$

$T < \infty$.

Proof: 1> $V_t, \tilde{V}_t$ admissible, adapted process.

2> Fundamental estimation: $\|(V_t - \tilde{V}_t) v \otimes e(0)\|^2 \le C_T \int_0^t \|(V_s - \tilde{V}_s) v \otimes e(0)\|^2 ds$

$\Rightarrow$ Gronwall ineq $\Rightarrow \|(V_t - \tilde{V}_t) v \otimes e(0)\|^2 = 0$.

H-P eq: $dU_t = (L_t dA_t^\dagger \boxed{- L_t^*} dA_t - \frac{1}{2} L_t L_t^* dt - i H_t dt) U_t$

$X_t = X_t \otimes \mathbb{1}$ bounded.

$dA_t e(0) = 0$.

$$d\tilde{V}_t = \Big(L_t\,(\underbrace{dA_t^\dagger + dA_t}_{dZ_t.}) - \tfrac{1}{2} L_t L_t^\dagger\,dt - iH_t\,dt\Big)\tilde{V}_t$$

$dZ_t \to$ B.M.

$\tilde{V}_t$: not bounded.

Principle idea for task:

Ref: · L'Accardi's Book, "Quantum theory and its stochastic limit" 2002, ch 6.

· van Handel thesis, 2007.

1> $Z_t \xrightarrow{\text{spec. thm}} \zeta_t = \iota(Z_t)$ B.M.

2> $d\tilde{V}_t = (L_t\,dZ_t + \cdots)\tilde{V}_t \xrightarrow{\text{spec. thm}} dV_t = (L_t\,d\zeta_t - \tfrac{1}{2}L_tL_t^\dagger\,dt - iH_t\,dt)V_t$

$L_t = L_t \otimes \mathbb{1}$

$\Downarrow$ matrix-valued Itô SDE.

3>. Coincide $\tilde{V}_t$ with $\iota^{-1}(V_t) \in M_n(\mathcal{N}(Z_t))$ on a dense subspace $\mathcal{H}_s \otimes \mathcal{E}' \subset \mathcal{H}_s \otimes \mathcal{E}$, contains all $v \otimes e(0)$. $v \in \mathcal{H}_s$.

$\mathcal{E}$: exp domain

4>. $\tilde{U}_t\, v \otimes e(0) = \tilde{V}_t\, v \otimes e(0) \xrightarrow{\text{spec. thm}} \boxed{\varphi(\tilde{U}_t^* X \tilde{U}_t) = \varphi(\tilde{V}_t^* \cdot X \cdot \tilde{V}_t)}$

$\boxed{\text{Theorem}}$: quantum Kallianpur–Striebel formula (B.F.).

$\forall X \in \mathcal{B}(\mathcal{H}_s)$, $\pi_t(X) = \tilde{U}_t^*\,\omega_t(X \otimes \mathbb{1} \mid Z_t)\,\tilde{U}_t$.

$$\omega_t(X \otimes \mathbb{1} \mid Z_t) = \mathbb{1} \otimes \frac{\varphi_s(V_t^* \cdot (X \otimes \mathbb{1}) \cdot V_t)}{\varphi_s(V_t^* V_t)}.$$

proof: $V_t^* V_t > 0$. $\Leftarrow$ V_t is invertible a.s.

Protter, "stochastic integration and D.E" 2005, pp. 326.

$\Rightarrow \quad V_t \longleftrightarrow \iota(V_t) \longrightarrow \iota(V_t)$

$\hookrightarrow$ driven z_t. $\qquad \hookrightarrow$ driven y_t.

Lemma: $d\bar{V}_t = (L_t\, dy_t - \frac{1}{2} L_t L_t^* dt - iH_t dt)\bar{V}_t$, $\bar{V}_0 = \mathbb{1}$. Ito SDE, $t \in [0,T]$

For all $X \in B(\mathcal{H}_S)$, define: $\bar{\sigma}_t(X) = \varphi_S(\bar{V}_t^* X \bar{V}_t)$,

$\bar{\pi}_t(X) := \frac{\bar{\sigma}_t(X)}{\bar{\sigma}_t(\mathbb{1})} = \frac{\varphi_S(\bar{V}_t^* X \bar{V}_t)}{\varphi_S(\bar{V}_t^* \bar{V}_t)}$.

Then: $\cdot\ \bar{\pi}_t(X) = \iota(\pi_t(X))$. Apply spe. thm. on $\Upsilon_T = U_T^* Z_T U_T$

$(\Omega, \mathcal{F}, \mu)$, ι, $\mathbb{P}$ induced by φ.

$\cdot\ y_t = \iota(Y_t)$ is B.M. under $\mathbb{Q}$, $d\mathbb{P} = \bar{\sigma}_T(\mathbb{1})\, d\mathbb{Q}$.

Proof: $\iota_Z : Z_T \longrightarrow L^\infty(\Omega_Z, \mathcal{F}_Z, \mu_Z)$

$\iota_Y : \Upsilon_T \longrightarrow L^\infty(\Omega_Y, \mathcal{F}_Y, \mu_Y)$

$\Rightarrow \iota : \Upsilon_T \longrightarrow L^\infty(\Omega_Z, \mathcal{F}_Z, \mu_Z)$

$\iota(X_Y) = \iota_Z(U_T X_Y U_T^*)$

$X_Y \in \Upsilon_T \Rightarrow U_T X_Y U_T^* \in Z_T$

Define proba. measure on $\mathcal{F}_Z$: $\mathbb{E}^{\mathbb{P}}(\iota(X_Y)) = \varphi(X_Y)$, $X_Y \in \Upsilon_T$

$\mathbb{E}^{\mathbb{Q}}(\iota_Z(X_Z)) = \varphi(X_Z)$, $X_Z \in Z_T$

$\Rightarrow y_t = \iota(Y_t) = \iota_Z(\underbrace{U_t Y_t U_t^*}_{Z_t}) = \iota_Z(Z_t) = z_t$ is B.M under $\mathbb{Q}$.

$\iota(\pi_t(X)) = \iota_Z(U_t \pi_t(X) U_t^*) = \iota_Z(\omega_t(X \otimes \mathbb{1} / Z_t))$

$\overset{K\text{-}S}{=} \iota_Z\left(\mathbb{1} \otimes \frac{\varphi_S(\hat{V}_t^* (X \otimes \mathbb{1}) \hat{V}_t)}{\varphi_S(\bar{V}_t^* \bar{V}_t)}\right) = \bar{\pi}_t(X)$.

· F_y : any functional of $\{y_t\}$

$$\mathbb{E}^{\mathbb{P}}(F_y) = \varphi(\iota^{-1}(F_y)) = \varphi\Big(\hat{U}_T^* \overbrace{\underbrace{U_T\, \iota^{-1}(F_y)\, U_T^*}_{\iota_Z^{-1}(F_y)}}^{\in Z_T} \hat{U}_T\Big)$$

$$= \varphi\big(V_T^* \,\hat{}\, \underbrace{\iota_Z^{-1}(F_y)}_{\mathbb{1}\otimes \iota_Z^{-1}(F_y)|_R} \,\hat{}\, V_T\big) \qquad V_T \in M_R(\mathcal{N}(Z_T)) \text{ commutative.}$$

$$= \mathbb{E}^{\mathbb{Q}}\big(\underbrace{\varphi_S(V_T^* V_T)}_{\bar\sigma_T(\mathbb{1}) \,=\, d\mathbb{P}/d\mathbb{Q}}\, F_y\big).$$

#,

Lemma: The innovation process: $W_t := y_t - \int_0^t \bar\pi_s(L_s + L_s^*)\, ds$ is B.M. under $\mathbb{P}$.

Proof: Classical Itô formula: $d\bar\sigma_t(\mathbb{1}) = \overset{\text{Tr}(\rho\,\cdot)}{\varphi_S}\big(d(V_t^* V_t)\big)$

$$= \bar\sigma_t(L_t + L_t^*)\, dy_t = \bar\pi_t(L_t + L_t^*)\, \bar\sigma_t(\mathbb{1})\, dy_t.$$

$\Rightarrow$ $\bar\sigma_t(\mathbb{1})$ is Girsanov transformation.

$\Rightarrow$ Girsanov thm: W_t is B.M under $\mathbb{P}$. #.

$\boxed{\text{Theorem}}$: quantum Kushner-Stratonovich eq:

$$\Rightarrow\ \boxed{d\bar\sigma_t(X) = \bar\sigma_t(\mathcal{L}_t^*(X))\, dt + \bar\sigma_t(L_t^* X + X L_t)\, dy_t}$$

innovation process.

$$\Rightarrow\ \boxed{d\bar\pi_t(X) = \bar\pi_t(\mathcal{L}_t^*(X))\, dt + \big(\bar\pi_t(L_t^* X + X L_t) - \bar\pi_t(L_t^* + L_t)\,\bar\pi_t(X)\big)\, dW_t}$$

$$\bar\sigma_0(X) = \bar\pi_0(X) = \varphi_S(X), \quad \mathcal{L}_t^*(X) = i[H_t, X] + L_t^* X L_t - \tfrac{1}{2} L_t^* L_t X - \tfrac{1}{2} X L_t^* L_t.$$

Coro: S.M.E.

Let ρ_t be random density matrix s.t. $\pi_t(X) = Tr(\rho_t X)$.

$$\boxed{d\rho_t = \mathcal{L}_t(\rho_t)dt + (L_t\rho_t + \rho_t L_t^* - Tr((L_t+L_t^*)\rho_t)\rho_t)(\underbrace{dy_t - Tr((L_t+L_t^*)\rho_t)dt}_{dW_t})}$$

$\mathcal{L}_t(\rho_t) = i[\rho, H_t] + L_t\rho L_t^* - \frac{1}{2}L_t^*L_t\rho - \frac{1}{2}\rho L_t^* L_t$, $\rho_0 = \rho$.

Coro: quantum Zakai eq:

Let ς_t be random s.a. matrix s.t. $\sigma_t(X) = Tr(\varsigma_t X)$

$$d\varsigma_t = \mathcal{L}_t(\varsigma_t)dt + (L_t\varsigma_t + \varsigma_t L_t^*)dy_t, \quad \varsigma_0 = \rho.$$

Zakai equation has unique solution $\Leftarrow$ Global Lip cond.

$\Rightarrow \rho_t = \varsigma_t / Tr(\varsigma_t)$ in $S_N = \{\rho \in \mathbb{C}^{N\times N} \mid \rho \geq 0, Tr(\rho) = 1, \rho = \rho^*\}$.

$\boxed{\text{Imperfect detection:}}$

* Add additional noise $(B_t + B_t^\dagger)$ to observation (in vacuum state).
* $\mathcal{H} = \mathcal{H}_S \otimes \Gamma_s(L^2(\mathbb{R}_+)) \otimes \Gamma_s(L^2(\mathbb{R}_+))$, $\varphi = \varphi_S \otimes \varphi_{R_1} \otimes \varphi_{R_2}$.

  $\tilde{U}_t = U_t \otimes \mathbb{1}$.

* $\tilde{j}_t(X) = \tilde{U}_t^*(X \otimes \mathbb{1} \otimes \mathbb{1})\tilde{U}_t$ does not change.

$$\tilde{Y}_t = \tilde{U}_t^*(\underset{\text{input}}{Z_t} + \underbrace{\overset{\geq 0}{\varepsilon}(B_t + B_t^\dagger)}_{\text{noise}})\tilde{U}_t = \underbrace{\tilde{U}_t^* Z_t \tilde{U}_t}_{Y_t} + \varepsilon(B_t + B_t^\dagger)$$

$\tilde{\Upsilon}_t = vN\{\tilde{Y}_s \mid 0 \le s \le t\}$ commutative.

Objective: $\varphi(\hat{j}_t(X) \mid \tilde{\Upsilon}_t)$.

Quantum conditional expectation $\xrightarrow{\text{spe. form}}$ Classical C.E.

$$\Rightarrow \varphi(\hat{j}_t(X) \mid \tilde{\Upsilon}_t) = \varphi(\varphi(\hat{j}_t(X) \mid vN\{\tilde{\Upsilon}_t, \Upsilon_t\}) \mid \tilde{\Upsilon}_t).$$

$B_t + B_t^\dagger$ is independent with Y_t, $j_t(X)$.

$$= \varphi(\overbrace{\varphi(\hat{j}_t(X) \mid \Upsilon_t)}^{\pi_t(X)} \mid \tilde{\Upsilon}_t)$$

$\Rightarrow$ $\pi_t(X)$ is not linear. $\Rightarrow$ filtering theory (B.F).

$\tilde{y}_t = \iota(\tilde{Y}_t)$, $\mathcal{F}_t^{\tilde{y}} = \sigma(\tilde{y}_t)$

$$\tilde{\pi}_t(X) := \boxed{\mathbb{E}^{\mathbb{P}}(\pi_t(X) \mid \mathcal{F}_t^{\tilde{y}})} \overset{B.F}{=} \frac{\mathbb{E}^{\mathbb{Q}}(\bar{\sigma}_t(X)\bar{\sigma}_t(\mathbb{1}) \mid \mathcal{F}_t^{\tilde{y}})}{\mathbb{E}^{\mathbb{Q}}(\bar{\sigma}_t(\mathbb{1}) \mid \mathcal{F}_t^{\tilde{y}})}.$$

$$d\bar{\sigma}_t(X) = \bar{\sigma}_t(\mathcal{L}_t^*(X))dt + \bar{\sigma}_t(L_t^* X + X L_t)\,dy_t.$$

$b_t = \iota(B_t + B_t^\dagger)$ B.M under $\mathbb{Q}$, indep. with y_t. y_t B.M. $\mathbb{Q}$.

$\Rightarrow \tilde{y}_t = y_t + \varepsilon b_t$.

indep. B.M

$$\underbrace{\frac{1}{\sqrt{1+\varepsilon^2}}\begin{bmatrix} 1 & \varepsilon \\ \varepsilon & -1 \end{bmatrix}}_{\text{orthogonal matrix}} \begin{bmatrix} y_t \\ b_t \end{bmatrix} = \frac{1}{\sqrt{1+\varepsilon^2}} \begin{bmatrix} \tilde{y}_t \\ \tilde{y}_t^\perp \end{bmatrix}$$

$\tilde{y}_t^\perp \Rightarrow$ independent with $\tilde{y}_t$.

ε (0, 1]

$\Rightarrow$ quad. variation: $(1+\varepsilon^2)t$.

$$d\bar{\sigma}_t(X) = \bar{\sigma}_t\left(\mathcal{L}_t^*(X)\right)dt + \overbrace{\frac{1}{1+\varepsilon^2}}^{\eta}\bar{\sigma}_t\left(L_t^* X + X L_t\right)d\tilde{y}_t + \frac{\varepsilon}{1+\varepsilon}\bar{\sigma}_t\left(L_t^* X + X L_t\right)d\tilde{y}_t^{\perp}$$

$$\Leftarrow \quad y_t = \frac{1}{1+\varepsilon^2}\left(\tilde{y}_t + \varepsilon\tilde{y}_t^{\perp}\right)$$

Ref: J. Xiong: An introduction to stochastic filtering theory. Lem 3.4.

Taking Condition expectation

$$d\mathbb{E}^{Q}\left(\bar{\sigma}_t(X)\,|\,\mathcal{F}_t^{\tilde{y}}\right) = \mathbb{E}^{Q}\left(\bar{\sigma}_t(\mathcal{L}_t^*(X))\,|\,\mathcal{F}_t^{\tilde{y}}\right)dt + \eta\,\mathbb{E}^{Q}\left(\bar{\sigma}_t(L^* X + X L)\,|\,\mathcal{F}_t^{\tilde{y}}\right)d\tilde{y}_t.$$

$$\Rightarrow \tilde{\bar{\sigma}}_t(X) = \mathbb{E}^{Q}\left(\bar{\sigma}_t(X)\,|\,\mathcal{F}_t^{\tilde{y}}\right).$$

$\Rightarrow$ Define: normalized observation process: $\hat{y}_t = \sqrt{\eta}\,\tilde{y}_t$, $\sqrt{\eta} = \sqrt{\frac{1}{1+\varepsilon^2}}$

$$\Rightarrow d\tilde{\bar{\sigma}}_t(X) = \tilde{\bar{\sigma}}_t\left(\mathcal{L}_t^*(X)\right)dt + \sqrt{\eta}\,\tilde{\bar{\sigma}}_t\left(L_t^* X + X L_t\right)d\hat{y}_t$$

$$\tilde{\bar{\sigma}}_0(X) = \rho_S(X)$$

$$\Rightarrow d\tilde{\pi}_t(X) = \tilde{\pi}_t\left(\mathcal{L}_t^*(X)\right)dt + \sqrt{\eta}\left(\tilde{\pi}_t(L_t^* X + X L_t) - \tilde{\pi}_t(L_t + L_t^*)\tilde{\pi}_t(X)\right)d\hat{W}_t$$

$$\hat{W}_t = \hat{y}_t - \sqrt{\eta}\int_0^t \tilde{\pi}_s(L_s + L_s^*)\,ds \quad \text{B.M.} \quad \mathbb{P}.$$

S.M.E.

[0,1]

$$\Rightarrow d\rho_t = \mathcal{L}_t(\rho_t)\,dt + \sqrt{\eta}\,\big(L_t\rho_t + \rho_t L_t^* - Tr\big((L_t + L_t^*)\rho_t\big)\rho_t\big)\,dW_t .$$

feedback law.

input $\rightarrow$ sys $\rightarrow$ output.

qubit. $\Rightarrow \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$

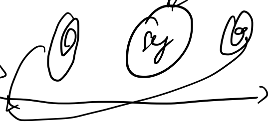
$\rho_0 = \begin{bmatrix} 1/2 & \\ & 1/2 \end{bmatrix}.$

$\rho_f \Rightarrow$ excited $\begin{bmatrix} 1 & \\ & 0 \end{bmatrix}.$

$$\mathcal{L}_t(\rho_t) = i[\rho_t, H_t] + L_t\rho_t L_t^* - \tfrac{1}{2}L_t^* L_t \rho_t - \tfrac{1}{2}\rho_t L_t^* L_t .$$

$H_0 + u_t H_1.$

field.



Eff., $u_t = u(\rho_t)$.

$u(y_t)$.