# Introduction to quantum feedback control. Lec 4. 05/11/2020

1> Input-output model for homodyne detection (H.D.).

2> Quantum conditional expectation and Bayes formula.

3> Quantum filtering theory.

## 1> Input-output model (H.D.)

H-P eq: $dU_t = (L_t dA_t^\dagger - L_t^* dA_t - \frac{1}{2} L_t L_t^* dt + iH_t dt) U_t$, $U_0 = \mathbb{1}$

$$L_t = \underbrace{L_t \otimes \mathbb{1}}_{\text{bounded process on } \mathcal{H}_S \text{ (finite-dim)}}, \quad U_t U_t^* = U_t^* U_t = \mathbb{1}$$

$\Rightarrow$ Evolution of sys: (H.P.) $\underbrace{j_t(X)}_{\text{quantum flow}} = U_t^* (X \otimes \mathbb{1}) U_t$.

Quantum Itô $\Rightarrow$ $dj_t(X) = j_t(\underbrace{\mathcal{L}^*(X)}_{\text{adjoint Lind.}}) dt + j_t([X, L_t]) dA_t^\dagger + j_t([L_t^*, X]) dA_t$

$\Rightarrow$ Evolution of field:

* $\underbrace{Z_t}_{\text{input noise}} = \underbrace{A_t + A_t^\dagger}_{\text{input field}}$ is essentially s.a. on $\underbrace{\mathcal{E}}_{\text{exp. domain}}$.

* $Z_t$ is Quantum B.M. (s.a. + commutative)

$$\Downarrow \text{spe. thm}$$

classical B.M.

* Observation process: $Y_t = U_t^* Z_t U_t$ (H.P.).

Ref: L'Accardi. Ch.4.26. 2002.

$Y_t = U_t^* Z_t U_t$.

Lem 1: $Z_t$ is Quantum B.M $\Rightarrow \mathcal{Z}_t := vN\{Z_s \mid 0 \le s \le t\}$.

$\Downarrow$

commutative.

Lem 2: (Self-nondemolition property)

$\Upsilon_t := vN\{Y_s \mid 0 \le s \le t\}$ is commutative $\forall t \in [0,T], T<\infty$.

$\Rightarrow Y_t \xrightarrow{\text{spc. thm}} y_t$ classical stochastic process.

ensures. $Y_t$ can be measured in a single realiz.

Proof: $E \in \mathcal{Z}_s$: projection in range of $\xi_{Z_s}$ ← spectral measure projection

Quantum Itô $\Rightarrow j_t(E) = U_t^* E U_t$

$$\Rightarrow \tilde{J}_t(E) = \tilde{J}_s(E) + \int_s^t \tilde{j}_r([L L^*, E])\, dA_r + \int_s^t \tilde{j}_r([E, L])\, dA_r^\dagger$$

adjoint Lind.

$$+ \int_s^t \tilde{J}_r(\mathcal{L}^*(E))\, dr$$

$E = \mathbb{1} \otimes E$, $L = L \otimes \mathbb{1}$

$\Rightarrow \tilde{J}_t(E) = \tilde{J}_s(E) \quad \forall\ 0 \le s \le t$.

Bounded f.c $\Rightarrow \xi_{Y_s} = U_s^* \xi_{Z_s} U_s \Rightarrow \Upsilon_t$ is commut. #

Lem 3: ( Non demolition prop )

$j_t(X) \in \Upsilon_t'$, $\forall X \in \mathcal{B}(\mathcal{H}_S)$, $t \in [0,T]$, $T < \infty$.

$\Rightarrow$ define joint statistic of sys events and observation

↳ conditional expectation.

Proof: $\Upsilon_t = U_t^* Z_t U_t$, $\hat{j}_t(X) = U_t^*(X \otimes \mathbb{1}) U_t$

$\mathcal{B}(\mathcal{H}_S) \otimes \mathbb{1}$ commutes with $Z_t = vN\{\mathbb{1} \otimes Z_s \mid 0 \le s \le t\}$.

$\Rightarrow \hat{j}_t(X) \in \Upsilon_t'$ #.

## $\boxed{\text{2>. Conditional expectation and Bayes formula.}}$

Classical filtering theory: $(\Omega, \mathcal{F}, \mathbb{P})$ : ref propobility app.

① $\mathbb{E}(x \mid \mathcal{G})$, $\mathcal{G} \subset \mathcal{F}$

② $\mathbb{E}\left(x - \mathbb{E}(x \mid \mathcal{G})\right)^2 = \min\limits_{y \in L^2(\mathcal{G})} \mathbb{E}(x-y)^2$

③ Bayes formula: $\mathbb{E}(x / \mathcal{G}) = \dfrac{\mathbb{E}^{\mathbb{Q}}(XM \mid \mathcal{G})}{\mathbb{E}^{\mathbb{Q}}(M \mid \mathcal{G})}$ $\quad M = d\mathbb{P}/d\mathbb{Q}$

④. define the ref proba $\Rightarrow$ Girsanov thm $M_t = d\mathbb{P}/d\mathbb{Q}_t$

⑤. Kallianpur - Striebel formula ( Bayes formula )

$$\mathbb{E}[f(x_t) \mid \underbrace{\mathcal{F}_t^Y}_{\sigma(Y_t)}] = \frac{\mathbb{E}^{Q_t}(M_t f(x_t) \mid \mathcal{F}_t^Y)}{\mathbb{E}^{Q_t}(M_t \mid \mathcal{F}_t^Y)}.$$

⑥ Kushner - Stratonovich eq: $\underline{\pi_t(f) = \mathbb{E}[\;|\;]}$

$(\mathcal{A}, \varphi)$

Restrictions on quantum Conditional exp: $\varphi(X \mid \mathcal{B})$, $\mathcal{B} \subset \mathcal{A}$

1> observation (conditional on) must be commutative.

2> conditional observable must commute with the observation.

self nondem

nondem.

Def (Quantum conditional expectation 1).

$(\mathcal{A}, \varphi)$, $\mathcal{B} \subset \mathcal{A}$ commutative. vN subalgebra. define map $\varphi(\cdot \mid \mathcal{B}) : \mathcal{B}' \longrightarrow \mathcal{B}$ : is called (a version of) quant. cond. exp. given $\mathcal{B}$. if $\varphi(\varphi(X \mid \mathcal{B}) S) = \varphi(XS)$, $\forall S \in \mathcal{B}$, $\forall X \in \mathcal{B}'$.

Lem 1: existence + uniqueness. $\varphi$-a.s.

Lem 2: (optimal estimation)

$$\forall X \in \mathcal{B}', \quad \| X - \varphi(X \mid \mathcal{B}) \|_\varphi = \min_{Y \in \mathcal{B}} \| X - Y \|_\varphi .$$

$$\varphi(A^* A) =: \|A\|_\varphi^2 .$$

Thm: Quantum Bayes formula 1.

$(\mathcal{A}, \varphi)$, $\mathcal{B} \subset \mathcal{A}$ commutative. Choose $V \in \mathcal{B}'$ (change-of-state operator), s.t. $V^* V \geq 0$. $\varphi(V^* V) = 1$. Then, define $\omega$ on $\mathcal{B}'$, $\omega(X) := \varphi(V^* X V)$

and $\omega(X \mid \mathcal{B}) = \dfrac{\varphi(V^* X V \mid \mathcal{B})}{\varphi(V^* V \mid \mathcal{B})}$.

Proof: $\varphi\Big(\varphi(V^* X V \mid \mathcal{B}) K\Big) = \cdots = \varphi\Big(\varphi(V^* V \mid \mathcal{B}) \omega(X \mid \mathcal{B}) K\Big)$

$K \in \mathcal{B}$

$$\Rightarrow \| \varphi(V^* X V \mid \mathcal{B}) - \varphi(V^* V \mid \mathcal{B}) \omega(X \mid \mathcal{B}) \|_\varphi^2 = 0 .$$

Def : (Quantum condition expectation 2) $\rightarrow$ affiliated to
$(\mathcal{A}, \varphi)$, $\mathcal{B} \subset \mathcal{A}$ commutative. Let $X$ of $\mathcal{B}'$ be s.a., suppose.
$\varphi(|X|) < \infty$. Then, any s.a. $\varphi(X|\mathcal{B})$ of $\mathcal{B}$ s.t.

$\varphi(\varphi(X|\mathcal{B}) \hat{\cdot} S) = \varphi(X \hat{\cdot} S)$ $\forall S \in \mathcal{B}$ is called (a version of) Quant. cond. exp.

Rem : $\mathcal{S}(vN(X, \mathcal{B})) :=$ all s.a. operator of $\overbrace{vN(X, \mathcal{B})}^{\text{commutative}}$.
$\hookrightarrow$ forms a commutative $*$-algebra (with $\mathbb{1}$) under $\hat{+}, \hat{\cdot}$.

- Existence + Uniq. $\Leftarrow$ Classical case + spectral.

Objective : Find Bayes formula for unbounded change-of-state op.

Setting for joint sys :

- $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$, $\dim(\mathcal{H}_S) = N < \infty$. $\mathcal{H}_R$ is separable.
- $\mathcal{A} = \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{B}(\mathcal{H}_R)$, $\varphi = \varphi_S \otimes \varphi_R \Rightarrow (\mathcal{A}, \varphi)$
- $\mathcal{B}_R \subset \mathcal{B}(\mathcal{H}_R)$ commutative $\Rightarrow \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{B}_R \subset (\mathbb{1} \otimes \mathcal{B}_R)'$
- $\mathcal{B}(\mathcal{H}_S) \cong M_N(\mathbb{C})$, $\varphi_S = Tr(\rho X_S)$, $X_S \in M_N(\mathbb{C})$.
  $\rho$: density op.

$\hookrightarrow$ $\mathcal{A} \cong M_N(\mathcal{B}(\mathcal{H}_R))$

$$\mathcal{A} \ni X = \begin{bmatrix} X_{11} & \cdots & X_{1N} \\ \vdots & \ddots & \vdots \\ X_{N1} & \cdots & X_{NN} \end{bmatrix} \quad X_{ij} \in \mathcal{B}(\mathcal{H}_R)$$

$$\Rightarrow \varphi(X) = Tr\left[\rho \begin{pmatrix} \varphi_R(X_{11}) & \cdots & \varphi(X_{1N}) \\ \vdots & \ddots & \vdots \\ \varphi(X_{N1}) & \cdots & \varphi_R(X_{NN}) \end{pmatrix}\right]$$

- define "partial" state of $\varphi$: $\forall X \in \mathcal{A}$ $\begin{cases} \varphi_R(X) \in M_N(\mathbb{C}) \\ \varphi_S(X) \in B(\mathcal{H}_R) \end{cases}$

Ref: Tomiyama "On tensor products of vN al" 1969.

- $\mathcal{N}(B_R) :=$ all normal operator affiliated $B_R$ ← commutative.

↳ forms commutative $*$-algebra under $\hat{+}, \hat{\cdot}$. ← Lec 2.

$\Longrightarrow M_N(\mathcal{N}(B_R))$ forms $*$-algebra under extension of $\hat{+}, \hat{\cdot}$.

Thm: quantum Bayes formula 2.

Let $V \in M_N(\mathcal{N}(B_R))$ (commutative) s.t. $\varphi(V^*V) = 1$. For $X \in B(\mathcal{H}_S) \otimes B_R$ define $\underline{\omega(X) = \varphi(V^* \hat{\cdot} X \hat{\cdot} V)}$. $\omega$ is a normal state on $B(\mathcal{H}_S) \otimes B_R$.

Moreover, if $V^*V > 0$, then

$$\omega(X \mid \mathbb{1} \otimes B_R) = \mathbb{1} \otimes \frac{\varphi_S(V^* \hat{\cdot} X \hat{\cdot} V)}{\varphi_S(V^*V)}, \quad \forall X$$

proof: ① well def $\omega$. ⇒ van Handel. thesis. pp. 107–108.

② $\varphi_R(\varphi_S(V^* \hat{\cdot} X \hat{\cdot} V) \hat{\cdot} K_R)$

$K = \mathbb{1} \otimes K_R \in \mathbb{1} \otimes B_R$.

$X \in B(\mathcal{H}_S) \otimes B_R$.

$$= \overbrace{\varphi_R(\varphi_S}^{\varphi}(V^* \hat{\cdot} X \cdot K \cdot V))$$

$$= \omega(XK) = \omega(\omega(X \mid \mathbb{1} \otimes B_R) K)$$

$$= \varphi_R(\varphi_S(V^* \hat{\cdot} \underbrace{\omega(X \mid \mathbb{1} \otimes B_R)}_{\mathbb{1} \otimes \omega(X \mid \mathbb{1} \otimes B_R)|_{B_R}} \hat{\cdot} K \hat{\cdot} V))$$

$$= \varphi_R(\varphi_S(V^*V) \hat{\cdot} \omega(X \mid \mathbb{1} \otimes B_R)|_{B_R} \cdot K_R)$$

#

## 3> Quantum filtering theory. $(\mathcal{A}, \varphi)$

H-pic.

Input field. $A_t$, $A_t^\dagger$ → Sys: $\hat{j}_t(X) = U_t^*(X \otimes \mathbb{1})U_t$ → Output field: $U_t^* A_t U_t$, $U_t^* A_t^* U_t$ → H. D. → Observation. $Y_t = U_t^*(Z_t)U_t$.

$Y_t$ → Filter → $\pi_t(X) = \varphi(\hat{j}_t(X) \mid \mathcal{Y}_t)$.