Lec 7: Feedback stabilization of angular momentum system.

SME: $$d\rho_t = -i[\omega J_z + u_t J_y, \rho_t]dt + M(J_z\rho_t J_z - J_z^2\rho_t/2 - \rho_t J_z^2/2)dt + \sqrt{\eta M}(J_z\rho_t + \rho_t J_z - 2\mathrm{Tr}(J_z\rho_t)\rho_t)dW_t.$$

$$= \boxed{F(\rho_t)dt} + \boxed{\sqrt{\eta}\,G(\rho_t)dW_t}$$

$$\rho \in S = \{\rho \in \mathbb{C}^{N\times N} \mid \rho = \rho^* \geq 0,\ \mathrm{Tr}(\rho) = 1\}$$

$$J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix} \quad J = \frac{N-1}{2}, \quad J_y = \begin{bmatrix} 0 & -ic_1 & & & \\ ic_1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & -ic_{2J} \\ & & & ic_{2J} & 0 \end{bmatrix} \quad c_m = \frac{\sqrt{(2J+1-m)m}}{2}$$

$\omega \geq 0$, $\eta \in (0,1]$, $M > 0$.

1> Asymptotic feedback stabilization

Ref: 1. R. van Handel, et al. IEEE TAC 2005

2. $\boxed{\text{M. Mirrahimi, R. van Handel, SIAM J contr. opt 2007}}$

1>. 2-dim.

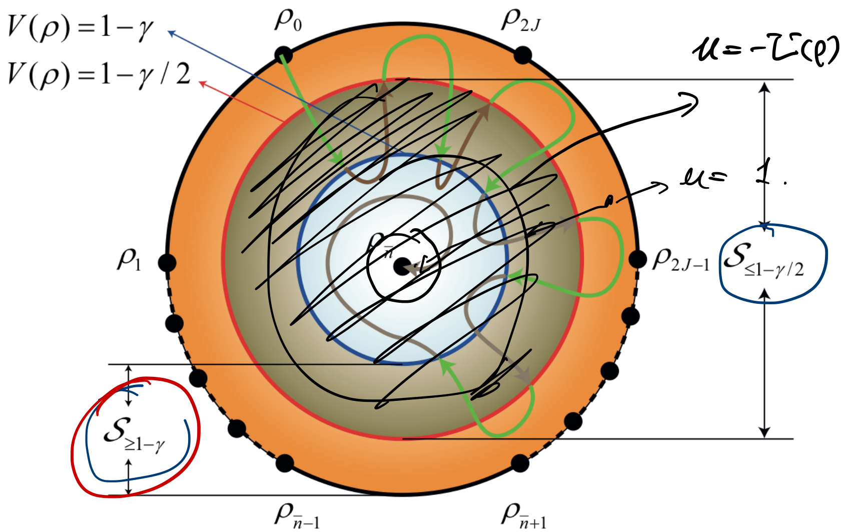
$$\rho = \frac{I + x\sigma_x + y\sigma_y + z\sigma_z}{2} = \begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

(x, y, z),

$$u_t = \alpha(1 + z_t) + \beta x_t. \qquad \bar{\rho} = \rho_g = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \longleftrightarrow (0,0,-1)$$

$$u(\rho_g) = 0, \quad u(\rho_e) \neq 0. \qquad \rho_e = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \longleftrightarrow (0,0,1)$$

$\Rightarrow$ Matlab, SOSTools. $\Rightarrow \boxed{V}$, $\mathcal{L}V \leq 0 \Rightarrow \rho_t \to \bar{\rho}$.



2>. N-dim. Spin-J.

$u\equiv 0$, $\bar{E}=\{\rho_0, \dots, \rho_{2J}\}$, $\rho_{\bar{n}}\in\bar{E}$ (target state), ρ_k antipodal eigenstate, $k\neq\bar{n}$.

$$\begin{bmatrix}1 & & \\ & 0 & \\ & & \ddots \\ & & & 0\end{bmatrix} \qquad \begin{bmatrix}0 & & \\ & \ddots & \\ & & 0 & \\ & & & 1\end{bmatrix}.$$

$\bar{V}(\rho) = 1 - \mathrm{Tr}(\rho\rho_{\bar{n}}) \in [0,1]$

$V(\rho)=0$ iff $\rho=\rho_{\bar{n}}$.

$V(\rho_k)=1$ $k\neq\bar{n}$.

$\gamma\in(0,1)$. $S_{\geq 1-\gamma} := \{\rho\in S \mid 1-\gamma \le \bar{V}(\rho) \le 1\}$

$S_{\le 1-\gamma/2} = \{\rho\in S \mid 0 \le \bar{V}(\rho) \le 1-\gamma/2\}$

Step 1: $\exists \gamma \in (0,1)$, s.t. $S_{\geq 1-\gamma}$ is not invariant under $u_t = 1$.

$\Longrightarrow$ probability of exiting $S_{\geq 1-\gamma}$ is >0.

Proof: Support theorem: (Stroock, Varadhan, 1972)

$$\underset{\cdot\, dW_t}{\text{Itô SDE}} \;=\!=\; \underset{\circ\, dW_t}{\text{Stra. SDE}} \;\longleftrightarrow\; \circ\, dW_t \leftrightarrow v_t \in \mathcal{V}$$

$\mathcal{V}$: all piecewise constant func, $\mathbb{R}_+ \to \mathbb{R}$.

$\to$ ODE.

$\Downarrow$

$\exists\, 0 < T < \infty$, $v_t \in \mathcal{V}$.
$\underline{\xi(T, x, v)} \in \underline{B_r(\bar{x})}$

$\Downarrow$

$\exists\, T < \infty$,
$\mathbb{P}(x_T \in B_r(\bar{x}) \mid x_0 = x) > 0$. $\longleftrightarrow$

Step 2: a.e. $\omega \in \Omega$, $\rho_t(\omega)$ can exit $S_{\geq 1-\gamma}$ under $u=1$ in $T_1(\omega) < \infty$.

Proof: Dynkin estimation.

$$\sup_{\rho_0 \in S_{\geq 1-\gamma}} \mathbb{E}_{\rho_0}[\tau] \leq \frac{T}{\underbrace{1 - \sup_{\rho_0 \in S_{\geq 1-\gamma}} \mathbb{P}_{\rho_0}(\tau \geq T)}} \leq \frac{T}{\xi} < \infty$$

τ: first exiting time from $S_{\geq 1-\gamma}$.

- $S_{\geq 1-\gamma}$ is compact.
- ρ_t is Feller continuous.

S is compact.

$$\exists\, \xi > 0, \quad \sup_{\rho_0 \in S_{\geq 1-\gamma}} \mathbb{P}_{\rho_0}(\tau \geq T) \leq 1 - \xi < 1$$

$\mathbb{P}_{\rho_0}(\tau < \infty) = 1.$

step 3: a.e. $\omega \in \Omega$, $\exists\, T_2(\omega) < \infty$, $\forall t \geq T_2(\omega)$, under a suitable feedback laws (u_t), $\rho_t(\omega)$ will stay in $S_{\leq 1-\gamma/2}$. and never exit.

Proof: $\bar{V}(\rho) = 1 - \mathrm{Tr}(\rho \rho_{\bar{n}})$

$u(\rho) = -\bar{\mathcal{I}}(\rho)$

$$\mathcal{L}\bar{V}(\rho) = u(\rho) \underbrace{\mathrm{Tr}(i[J_y, \rho]\rho_{\bar{n}})}_{\bar{\mathcal{I}}(\rho)} \leq -\bar{\mathcal{I}}^2(\rho) \leq 0$$

$\bar{\mathcal{I}}(\rho_k) = 0 \quad \forall \rho_k \in \bar{E}.$

$\rho \in S_{\leq 1-\gamma/2}.\ (\ni \rho_{\bar{n}})$, $\quad \underline{u(\rho) = -\bar{\mathcal{I}}(\rho).}$

τ first exiting time from $S_{\leq 1-\gamma/2}$.

$$\mathbb{E}[\bar{V}(\rho_{\tau\wedge t})] = \bar{V}(\rho_0) + \underbrace{\mathbb{E}\left[\int_0^{\tau\wedge t} \mathcal{L}\bar{V}(\rho_s)\,ds\right]}_{\leq 0}$$

$$\leq \bar{V}(\rho_0)$$

$(\underline{\rho_0 \in S_{\leq 1-\gamma}})$.

$\Rightarrow$ Supermartingale ineq:

$$\mathbb{P}\left(\sup_{t\geq 0} \bar{V}(\rho_t) \geq 1-\gamma/2\right) = \mathbb{P}\left(\sup_{t\geq 0} \bar{V}(\rho_{t\wedge\tau}) \geq 1-\gamma/2\right)$$

$$\leq \frac{\bar{V}(\rho_0)}{1-\gamma/2} \leq \frac{1-\gamma}{1-\gamma/2} < 1.$$

$S_{\geq 1-\gamma/2}$ n time.

- Strong Markov property.
- Borel Cantelli lemma

$\Big\} \Rightarrow \mathbb{P}(\tau_n^{(\geq 1-\gamma/2)} < \infty$, for infi many $n) = 0.$

$\Uparrow$

$\sum \mathbb{P}(\tau_n^{\geq 1-\gamma/2} < \infty) < \infty.$

Step: $\forall \rho_0 \in S$, $\rho_t \xrightarrow{a.s} \rho_{\bar n}$ under a suitable feedback laws.

Proof: $\tilde{V}(\rho) = 1 - \mathrm{Tr}(\rho\rho_{\bar n}^2) \in [0,1] \Rightarrow$ sto LaSalle thm

$\Downarrow$

$\lim \mathcal{L}\tilde{V} = 0$

$\Downarrow$

$\rho_t \to \rho_{\bar n}$ a.s.

Thm (M.M, R.v.H 2007)

- $u(\rho) = -\mathcal{I}(\rho)$ for $\rho \in S_{<1-\gamma}$
- $u(\rho) = 1$ for $\rho \in S_{\geq 1-\gamma/2}$.
- for $\rho \in \mathcal{B} = \{\gamma/2 < \mathrm{Tr}(\rho\rho_{\bar n}) < \gamma\}$, then $u(\rho) = -\mathcal{I}(\rho)$, if ρ_t enter $\mathcal{B}$ through $\tilde{V}(\rho) = 1-\gamma$, and $u(\rho) = 1$ otherwise.

Then, $\rho_t \xrightarrow{a.s} \rho_{\bar n}$.

Improvements:

1. step 1, 2 $\longrightarrow$ arbitrary $\gamma > 0$.
2. step 4 $\longrightarrow$ local Lyapunov argument, estimate the rate of convergence.
3. Provide the sufficient conditions on $u(\rho)$, $\rho_t \xrightarrow[a.s]{exp} \rho_{\bar n}$.

2> Exponential feedback stabilization.

Ref: Liang, N. Amini, P. Mason, Siam J. Control, opt, 2019.

① Preliminary results (without proof).

Lemma: Assume $u \in C^1(S, \mathbb{R})$. The set of all positive-definite matrices is a.s. invariant for SME.

$$\mathrm{int}(S) := \{\rho \in S \mid \rho > 0\}$$

Lemma: $\eta = 1$, $u \in C^1$, $\partial S := \{\rho \in S \mid \det(\rho) = 0\}$ is invariant.

Prop: $\eta \in (0,1)$, $u \in C^1$, $u(\rho_{\bar{n}}) = 0$. $\nabla u \cdot G(\rho_0) \neq 0$ or $\nabla u \cdot F(\rho_0) \neq 0$ for any $\rho_0 \in \{\rho \in S \setminus \rho_{\bar{n}} \mid \rho_{k,k} = 0$ for some k and $u(\rho) = 0\}$.

Then, $\forall \rho_0 \in \{\rho \in S \setminus \rho_{\bar{n}} \mid \rho_{k,k} = 0$ for some $k\}$, $\forall \varepsilon > 0$, $\exists$ at most one $\rho_v(t)$ starting from ρ_0 which lies in ∂S. For any other $\rho_0 \in \partial S \setminus \rho_{\bar{n}}$, $\forall v_t \in V$, $\rho_v(t) > 0$ for $t > 0$.

$$\begin{bmatrix} \rho_{00} & & \\ & \ddots & \\ & & \rho_{J,J} \end{bmatrix}$$

$$V = \sum \sqrt{\rho_{n,n}}$$

$$d\rho = F(\rho)\,dt + \sqrt{\eta}\, G(\rho)\, dW_t. \quad \text{SME}$$

$$\frac{d\rho_v}{dt} = \hat{F}_u(\rho) + \sqrt{\eta}\, G(\rho) v_t. \quad \text{ODE} \qquad v_t \in V.$$

Pb: $\eta \in (0,1)$ cond. on u (fix.) $\rho(t) > 0$ for $t > 0$, a.s. for all $\rho_0 \in \partial S \setminus \rho_n$?

$$J_\gamma = \begin{bmatrix} J & & & & \\ & \ddots & & & \\ & & J-n & & \\ & & & \ddots & \\ & & & & -J \end{bmatrix}$$

$$J_\gamma \cdot \rho_{\bar n} = (J-\bar n)\,\rho_{\bar n}$$

1. Reachability of ODE.

Lemma. $u(\rho_{\bar n}) = 0$, $u \in C^1$. Suppose $\exists\, v \in V$ s.t. for any $\rho \in \{\rho_{k,k} = 0 \text{ for some } k,\ u(\rho)=0\}$, $u(\rho_v(t)) \neq 0$ for $t \in (0,\varepsilon)$ for $\varepsilon < \infty$.

Assume, $\forall \rho \in \{J - \bar n - \mathrm{Tr}(J_\gamma \rho) = 0\} \setminus \rho_{\bar n}$.

$$2\eta M\, \mathrm{Var}(\rho)\, \rho_{\bar n,\bar n} > u(\rho)\, \mathrm{Tr}(i[J_\gamma, \rho]\rho_{\bar n}).$$

$$= \mathrm{Tr}(J_\gamma^2 \rho) - \mathrm{Tr}(J_\gamma \rho)^2 \overset{C.S.}{\geq} 0.$$

$\mathrm{Var}(\rho) = 0$ iff $\rho \in \bar E = \{\rho_0, \dots, \rho_{2J}\}$

$$\begin{bmatrix} \rho_{22} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \rho_{J2} \end{bmatrix}$$

Then, $\forall \rho_0 \in S$, and $\forall r > 0$, $\exists\, T \in (0,\infty)$ s.t. $\rho_v(T) \in B_r(\rho_{\bar n})$.

$$v_t = \frac{K}{P_{\bar n}\, \rho_{\bar n,\bar n}}$$

Proof: $\mathrm{Tr}(\rho\rho_{\bar n}) = 1 \iff \rho = \rho_{\bar n}$.

$\mathrm{Tr}(\rho\rho_{\bar n}) = \rho_{\bar n,\bar n}$.

$|\Delta_{\bar n}| < R$.

ODE. $\downarrow$?

$$(\dot\rho_v(t))_{\bar n,\bar n} = \Delta_{\bar n}(\rho_v) + 2\sqrt{\eta} M\, P_{\bar n}(\rho_v)\, (\rho_v)_{\bar n\bar n}\, v_t.$$

$$\Delta_{\bar n}(\rho) = 2\eta M\left(\mathrm{Tr}(J_\gamma^2\rho) - (J-\bar n)^2\right)\rho_{\bar n,\bar n} - u(\rho)\,\mathrm{Tr}(i[J_\gamma,\rho]\rho_{\bar n}) + 4\eta M\, P_{\bar n}(\rho)\,\mathrm{Tr}(\rho J_\gamma)\,\rho_{\bar n,\bar n}.$$

$$P_{\bar n}(\rho) = J - \bar n - \mathrm{Tr}(J_\gamma \rho)$$

$P_{\bar n}(\rho) = 0$
$\Updownarrow$
$J - \bar n = \mathrm{Tr}(J_\gamma \rho)$

$P_{\bar n}(\rho) = J - \bar n - \mathrm{Tr}(J_z \rho)$

① . assumption $\Rightarrow \exists\, v,\ u(\rho) \neq 0 \xRightarrow{\text{Prop 3}} \rho_{\bar n,\bar n}(q) > 0.$

$\Rightarrow$ w.l.o.g suppose $\underline{\rho_{\bar n,\bar n}(0) > 0.}$

② . 1. $\bar n = 0$, $P_{\bar n}(\rho) = J - \mathrm{Tr}(\rho J_z) = \sum_{n=0}^{2J} n \cdot \rho_{n,n} \geq 0.$, $P_0(\rho) = 0$ iff $\rho = \rho_0$ $\left[\begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{smallmatrix}\right]$

$\bar n = 2J$, $P_{\bar n}(\rho) = -J - \mathrm{Tr}(\rho J_z) = -2J + \sum_{n=0}^{2J} n\, \rho_{n,n} \leq 0.$, $P_{2J}(\rho) = 0$ iff $\rho = \rho_{2J}$ $\left[\begin{smallmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{smallmatrix}\right]$

$\left\{ \begin{array}{l} 0 \leq \rho_{n,n} \leq 1 \\ \sum_{n=0}^{2J} \rho_{n,n} = 1 \end{array} \right.$

$\Rightarrow v = \dfrac{K}{P_{\bar n}(\rho)\, \rho_{\bar n,\bar n}}$, K sufficiently large.

$\Rightarrow \forall r > 0,\ \exists T,\ \rho_r(T) \in B_r(\rho_{\bar n})$.

2. $\bar n \in \{1, \ldots, 2J-1\}$.

$P_{\bar n}(\rho) \neq 0.$

$P_{\bar n}(\rho) = 0$? $\Rightarrow (\dot\rho_{r}(t))_{\bar n \bar n} = \Delta_{\bar n}(\rho) = 2\eta M\, \mathrm{Var}(\rho)\, \rho_{\bar n \bar n} - u(\rho)$, $\mathrm{Tr}(i[J_y, \rho]\rho_{\bar n})$

$> 0.$ ↑ assumption.

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③ Reachability of SME.

$\inf\{t \geq 0 \mid \rho_t \in \underline{B_r(\rho_{\bar n})}\}$

Dynkin estimation.

$\rho_t \to$ Feller continuous

$\underline{S \setminus B_r(\rho_{\bar n})}$ is compact

$\Bigg\}\ \mathbb{P}_{\rho_0}(\tau_r < \infty) = 1.$