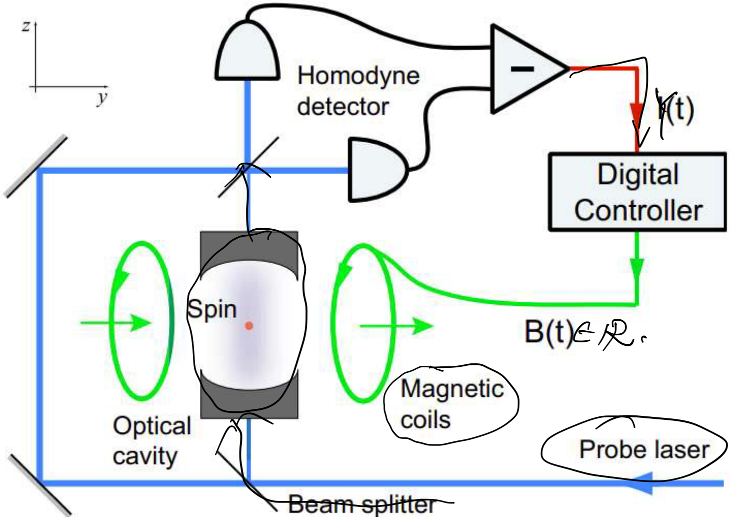
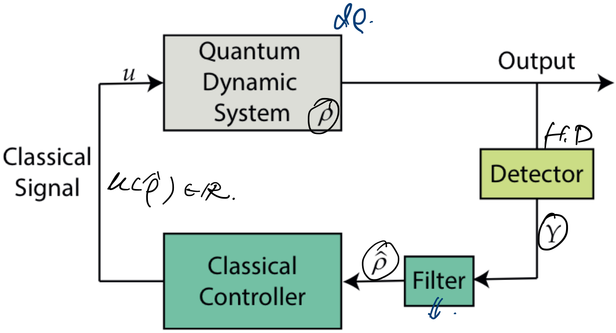
Feedback stabilization of open spin system. 19/Nov/2020

(measurement-based feedback control)





Angular momentum system (spin system): a quantum system (dim = N) with fixed angular momentum $J = \frac{N-1}{2}$, $2J \in \mathbb{N}$.

Control objective:

Ex: an atom (spin-½ system)

$$\Rightarrow \varphi(j_t(H_0)) \xrightarrow{u_t} \begin{cases} \hbar\omega_0/2 = Tr(H_0 \rho_e) \\ -\hbar\omega_0/2 = Tr(H_0 \rho_g) \end{cases}$$

(energy)

$$\varphi(j_t(H_0)) = \varphi(\pi_t(H_0)) = Tr(\rho_t H_0).$$

$$H_0 = \begin{bmatrix} \hbar\omega_0/2 & \\ & -\hbar\omega_0/2 \end{bmatrix} = \frac{\hbar\omega_0}{2} \underbrace{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}_{\sigma_z}.$$

$$\rho_e = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} \quad \rho_g = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$$

Goal: $\rho_t \xrightarrow{u_t} \bar{\rho}$ (target pure state)

$\uparrow$ conditional density matrix.

(Quantum state preparation).

N-level A.A. system.

$$u_t = f(\tau)$$

Lindblad $F(\rho)$

S.M.E. $d\rho_t = -i[\omega J_z + u_t J_y, \rho_t]dt + M(J_z\rho J_z - J_z^2\rho_t/2 - \rho_t J_z^2/2)dt$

~~$+\sqrt{\eta M}(J_z\rho_t + \rho_t J_z - 2\,Tr(J_z\rho_t)\rho_t)dW_t$~~

back action of measurement. $G(\rho)$

state space

Ref: R. van Handel, et al. J. opt. B, 2005.

- $\rho \in S_N = \{\rho \in \mathbb{C}^{N\times N} \mid \rho = \rho^* \geq 0,\ Tr(\rho) = 1\}$
- W_t is 1-dim B.M. in $(\Omega, \mathcal{F}, \mathbb{P})$
- $J_z e_n = (J-n)e_n$, $n \in \{0, \dots, 2J\}$, $\{e_0, \dots, e_{2J}\}$ orthonormal basis of $\mathbb{C}^N$

$$J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix} \qquad e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad e_{2J} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

- $J_y e_n = -i c_n e_{n-1} + i c_{n+1} e_{n+1}$, $n \in \{0, \dots, 2J\}$

$$J_y = \begin{bmatrix} 0 & -ic_1 & 0 & \cdots & 0 \\ ic_1 & 0 & -ic_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & ic_{2J-1} & 0 & -ic_{2J} \\ 0 & \cdots & 0 & ic_{2J} & 0 \end{bmatrix} \qquad c_m = \frac{1}{2}\sqrt{(2J+1-m)m}$$

- $\eta \in (0,1]$: efficiency of detectors

  $M > 0$: strength of interaction between atoms and probe

  $\omega \geq 0$: corresponding to free Hamiltonian

H1: $\omega = \hat{\omega}$, $M = \hat{M}$, $\eta = \hat{\eta}$, $\rho_0 = \hat{\rho}_0 \Longrightarrow \rho_t = \hat{\rho}_t$, $\forall t \geq 0$ a.s.

(ω ← sys, $\hat{\omega}$ ← filter)

1. Stochastic stability: Ref: 1. R. Khasminskii, "Stochastic stability of D.E"

2. X. Mao "SDEs and applications" 2007.

Itô's formula:

$$dq_t = f(q_t)\,dt + g(q_t)\,dW_t, \quad q_t \in Q \subset \mathbb{R}^p,$$

!!! $C^2 \ni V : Q \times \mathbb{R}_+ \longrightarrow \mathbb{R}$.

$$dV(q,t) = \underbrace{\mathcal{L}V(q,t)}_{\text{infinitesimal generator}}\,dt + \sum_{i=1}^{p} \frac{\partial V(q,t)}{\partial q_i} g_i(q)\,dW_t.$$

Bures metric: $\rho_a, \rho_b \in S_N$.

$$d(\rho_a, \rho_b) = \sqrt{2 - 2\,\mathrm{Tr}\sqrt{\sqrt{\rho_b}\,\rho_a\sqrt{\rho_b}}}.$$

$$\rho_b = \psi\psi^* \Rightarrow d(\rho_a, \rho_b) = \sqrt{2 - 2\sqrt{\mathrm{Tr}(\rho_a \rho_b)}}$$

$$d(\rho_a, E) = \inf_{\rho \in E} d(\rho_a, \rho), \quad E \subseteq S_N$$

$\Downarrow$.

$$B_r(E) = \{\rho \in S_N \mid d(\rho, E) < r\}.$$

Def (stability):

$\bar{E} \subseteq S_N$ is an invariant subset for SME, $\bar{E}$ is called

<1> locally stable in probability: $\forall \varepsilon \in (0,1)$, $r > 0$, $\exists \delta = \delta(\varepsilon, r)$ s.t.

$$\mathbb{P}(\rho_t \in B_r(\bar{E}), \text{ for } t \geq 0) \geq 1 - \varepsilon, \quad \forall \rho_0 \in B_\delta(\bar{E}).$$

<2> a.s. asymptotically stable, if it is locally stable in probab. and

$$\mathbb{P}\left(\lim_{t\to\infty} d(\rho_t, \bar{E}) = 0\right) = 1, \quad \forall \rho_0 \in S_N.$$

<3> exponentially stable in mean:

(constant)

$$\mathbb{E}\left[d(\rho_t, \bar{E}) \right] \leq \alpha \cdot d(\rho_0, \bar{E}) \cdot e^{-\beta t} \quad \forall \rho_0 \in S_N.$$

smallest value of $-\beta$: average Lyapunov exp.

<4> a.s. exponentially stable:

$$\underbrace{\limsup_{t\to\infty} \frac{1}{t} \log d(\rho_t, \bar{E})}_{\text{sample Lyapunov exponent}} < 0 \quad \text{a.s.} \quad \forall \rho_0 \in S_N.$$

2. Quantum state reduction.

1>. Purification: which roles of $F(\rho)$ and $G(\rho)$ of SME play on the preparation of the pure states?

$\mathrm{Tr}(\rho^2) = 1 \Rightarrow \rho$ is pure.

$$S(\rho) = 1 - \mathrm{Tr}(\rho^2) \Longrightarrow \underbrace{\mathcal{L}S(\rho)}_{\leq 0} = 2M\left(\overbrace{\mathrm{Tr}(\rho^2 J_z^2) - \mathrm{Tr}(\rho J_z \rho J_z)}^{\text{C-S} \Rightarrow \geq 0}\right)$$

① $\mathcal{L}S(\rho)$ independent of u_t.

② $F(\rho)$. negative effect.

③ $G(\rho)$ positive effect.

$$\underbrace{-\int \underbrace{\mathrm{Tr}(G^2(\rho))}_{\geq 0}}_{\leq 0} \leftarrow \text{backaction measu...}$$

2>. $u_t \equiv 0$. $\bar{E} = \{\rho_0, \dots, \rho_{2J}\}$. equilibria.

$$\rho_0 = \begin{bmatrix} 1 & & \\ & 0 & \\ & & \ddots \\ & & & 0 \end{bmatrix} \quad \cdots \quad \rho_{2J} = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ & & & 1 \end{bmatrix}$$

variance function

$$V(\rho) = \mathrm{Tr}(J_z^2 \rho) - \mathrm{Tr}(J_z \rho)^2 \overset{C.S}{\geq} 0. \qquad V(\rho) = 0 \text{ iff } \rho \in \bar{E}.$$

$$\mathcal{L}V(\rho) = -\mathrm{Tr}(J_z G(\rho))^2 = -4\underbrace{M\eta}_{>0} V(\rho)^2 \leq 0. \qquad \mathcal{L}V(\rho) = 0 \text{ iff } \rho \in \bar{E}.$$

Thm: (Stochastic LaSalle thm) Ref: X. Mao. "Stochastic LaSalle thm" 1999.

$$\mathcal{L}V(\rho) \leq 0 \Longrightarrow \lim_{t\to\infty} \mathcal{L}V(\rho_t) = 0 \quad a.s.$$

$$\Longrightarrow \rho_t \xrightarrow{u_t \equiv 0} \bar{E} = \{\rho_0, \dots, \rho_{2J}\} \ni \rho_n.$$

- Probability of convergence to $\bar{\rho} \in \bar{E}$.

$$\mathcal{L}\,\mathrm{Tr}(\rho_t \rho_n) = 0 \Longrightarrow \mathrm{Tr}(\rho_t \rho_n) \text{ is a martingale.}$$

$$\downarrow$$

$$\mathbb{1}_{\{\rho_t \to \rho_n\}}$$

$$\mathbb{P}(\rho_t \to \rho_n) = \lim_{t\to\infty} \mathbb{E}[\mathrm{Tr}(\rho_t \rho_n)] = \boxed{\mathrm{Tr}(\rho_0 \rho_n)}$$

$\rho_t \xrightarrow{u_t = 0} \rho_0 \quad (\mathbb{P} = \mathrm{Tr}(\rho_0 \rho_0))$

$\rho_t \longrightarrow \rho_1 \quad (\mathbb{P} = \mathrm{Tr}(\rho_0 \rho_1))$

$\vdots$

$\rho_t \longrightarrow \rho_{2J} \quad (\mathbb{P} = \mathrm{Tr}(\rho_0 \rho_{2J}))$

Goal: $\rho_{\bar n} \in \bar E$. $\quad \rho_t \xrightarrow{u_t} \rho_{\bar n} \in \bar E$.

=> Exponential Q.S.R.

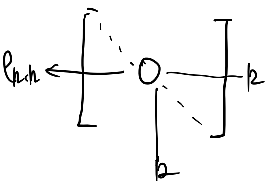
Ref: · Liang, et al. 2019 SIAM J. Control Opt.

· Tristant, et al. 2014 Commu. Math. Phys.

Lemma: $u \equiv 0$. * If $\rho_{k,k}(0) = 0$, $\mathbb{P}\{\rho_{k,k}(t) = 0, \forall t \geq 0\} = 1$.

* If $\rho_{k,k}(0) \neq 0$, $\mathbb{P}\{\rho_{k,k}(t) \neq 0, \forall t \geq 0\} = 1$.

$\rho_{k,k} := \mathrm{Tr}(\rho \cdot \rho_k)$



Proof: * $d\rho_{k,k}(t) = \sqrt{\eta}\mu\,(J - k - \mathrm{Tr}(J_z \rho_t))\,\rho_{k,k}(t)\,dW_t$. ✓.

$\rho_t \in S_N$ ⇑ compact space. $\quad \leq R > 0$. ↑ constant.

* Suppose $\rho_{k,k}(0) > 0$, $\mathbb{P}\{\rho_{k,k}(t) \neq 0, \forall t \geq 0\} < 1$.

$\tau = \inf\{t \geq 0 \mid \rho_{k,k}(t) = 0\}$

⇓

$\mathbb{P}(\tau < \infty) > 0 \Rightarrow \mathbb{P}(\tau \leq T) > 0$.

$\varepsilon \in (0, \rho_{k,k}(0))$, $V(\rho) = \frac{1}{\rho_{k,k}}$, $\rho_{k,k} \geq \varepsilon$.

If $\rho_{k,k} \geq \varepsilon$, $\quad \mathcal{L}V(\rho) = \dfrac{\eta\,(G(\rho))^2_{k,k}}{\rho^3_{k,k}} \leq R^2 V(\rho)$

Define: $f(\rho, t) = e^{-R^2 t} V(\rho)$

⇓

$\mathcal{L}f(\rho, t) = e^{-R^2 t}(-R^2 V(\rho) + \mathcal{L}V(\rho)) \leq 0$.

$\tau_\varepsilon = \inf\{t \geq 0 \mid \rho_{k,k}(t) \notin (\varepsilon, 1]\}$.

Itô's formula: $\mathbb{E}[f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T)] = \bar{V}_0 + \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge T} \mathcal{L} f(\rho_s, s)\, ds\right]$

$\leq \bar{V}_0 = \frac{1}{\rho_{kk}(0)}$ (the integral term ≤ 0)

$\Rightarrow \tau_\varepsilon \leq \tau$ ($\tau_\varepsilon \nearrow$, $\varepsilon \nearrow 0$) $\Rightarrow$ conditioning to $\{\tau < T\}$.

$f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T) = f(\rho_{\tau_\varepsilon}, \tau_\varepsilon) = e^{-R^2 T} \varepsilon^{-1}$.

$\mathbb{E}[e^{-R^2 T} \varepsilon^{-1} \mathbb{1}_{\{\tau \leq T\}}] = \mathbb{E}[f(\rho_{\tau_\varepsilon}, \tau_\varepsilon)\mathbb{1}_{\{\tau \leq T\}}]$

$\leq \mathbb{E}[f(\rho_{\tau_\varepsilon \wedge T}, \tau_\varepsilon \wedge T)] \leq \frac{1}{\rho_{kk}(0)}$

$\Rightarrow \mathbb{P}(\tau \leq T) \leq \varepsilon \cdot \frac{e^{R^2 T}}{\rho_{k,k}(0)} \xrightarrow{\varepsilon \to 0} \mathbb{P}(\tau \leq T) = 0 \Rightarrow$ contradiction

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Thm: (exp. Q.S.R.)

$u_t \equiv 0$, $\bar{E} = \{\rho_0, \dots, \rho_{2J}\}$ is exp stable in mean and a.s. with Lyapunov exponent $\leq -g\mu/2$.

Proof: $I = \{k \mid \rho_{k,k}(0) = 0\}$. $S_I = \{\rho \in S_N \mid \rho_{kk} = 0 \;\; \forall\, k \in I\}$.

(S_I:) invariant for SME ($u \equiv 0$)

$\Rightarrow \bar{V}(\rho) = \frac{1}{2}\sum_{\substack{n \neq m \\ n,m=0}}^{2J} \sqrt{\mathrm{Tr}(\rho P_n)\mathrm{Tr}(\rho P_m)} = \frac{1}{2}\sum_{n\neq m}\sqrt{\rho_{n,n}\rho_{m,m}} \geq 0$

$V(\rho) = 0$ iff $\rho \in \bar{E}$.

$$\mathcal{L}\bar{V}(\rho) \le -\frac{\eta M}{2}\bar{V}(\rho).$$

↳ [Itô's formula] ⇒ $\mathbb{E}[\bar{V}(\rho_t)] = \bar{V}(\rho_0) + \mathbb{E}\left[\int_0^t [\mathcal{L}\bar{V}(\rho_s)]ds\right]$

$$\le \bar{V}(\rho_0) - \frac{\eta M}{2}\int_0^t \mathbb{E}[\bar{V}(\rho_s)]ds.$$

Grönwall ineq:

$$\mathbb{E}[\bar{V}(\rho_t)] \le \bar{V}(\rho_0)\, e^{-\eta M/2\, t}.$$

$$c_1 d(\rho,\bar{E}) \le \bar{V}(\rho) \le c_2 d(\rho,\bar{E})$$

$c_1 = 1/2$, $c_2 = \eta(2\eta+1)$

$$\Rightarrow \mathbb{E}[d(\rho_t,\bar{E})] \le \frac{c_2}{c_1} d(\rho_0,\bar{E})\, e^{-\eta M/2\, t}, \quad \forall \rho_0 \in S_N.$$

* $Q(\rho_t, t) = e^{\eta M/2\, t}\,\bar{V}(\rho_t) \ge 0.$

$$\mathcal{L}Q(\rho_t,t) = e^{\eta M/2\, t}\left(\frac{\eta M}{2}\bar{V}(\rho) + \mathcal{L}\bar{V}(\rho)\right) \le 0$$

≤ 0

⇒ $Q(\rho_t,t)$ is positive supermartingale.

Doob's martingale convergence thm: $Q(\rho_t, t) \xrightarrow{t\to\infty} A < \infty$ a.s. (bounded r.v a.s.)

$A = \sup_{t\ge 0} Q(\rho_t, t)$.

$$\Rightarrow \sup_{t\ge 0} \bar{V}(\rho_t) = A\, e^{-\eta M/t} \quad \text{a.s.}$$

$$\Rightarrow \limsup_{t\to\infty} \frac{1}{t}\log \bar{V}(\rho_t) \le -\eta M/2, \quad c_1 d(\rho,\bar{E}) \le \bar{V}(\rho) < c_2 d(\rho,\bar{E})$$

$$\Rightarrow \limsup_{t\to\infty} \frac{1}{t}\log d(\rho_t,\bar{E}) \le -\eta M/2.$$

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