Introduction to quantum feedback control

CY-McGill Mathematical Physics Weekly Seminar

Weichao LIANG

weichao.liang@u-cergy.fr

Lab AGM/Dpt mathématiques, CY Cergy Paris Université

November 5, 2020

Outline

Input-output model for homodyne detection

Input-output model for homodyne detection

References :

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus*", 1992.
2. A. Barchielli, "*Continual measurements in quantum mechanics and quantum stochastic calculus*", In Open Quantum Systems III (pp. 207-292), 2006.
3. L. Accardi, Y. G. Lu, I. Volovich, "*Quantum theory and its stochastic limit, Ch 4-6*", Springer, 2002.
4. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering*", SIAM. J. Control Optim, 2007.
5. L. Bouten, R. van Handel, "*On the separation principle in quantum control*", Quantum Stoch. Inf. (pp. 206-238), 2008.

Input-output model for homodyne detection [1] [2]

H-P equation : $dU_t = \left(L_t dA_t^\dagger - L_t^* dA_t - \frac{1}{2} L_t L_t^* dt + i H_t dt\right) U_t, \quad U_0 = \mathbb{1}.$

Evolution of system :

- System observable (H.P.) : $j_t(X) = U_t^*(X \otimes \mathbb{1}) U_t$
- $dj_t(X) = j_t(\mathcal{L}^*(X))dt + j_t([X, L_t])dA_t^\dagger + j_t([L_t^*, X])dA_t$

Evolution of field :

- $Z_t := A_t + A_t^\dagger$ is essentially selfadjoint on $\mathcal{E}$
- $\bar{Z}_t$ is quantum Brownian motion (selfadjoint+commutative)
- Observation process (H.P.) : $Y_t = U_t^* Z_t U_t$

1. L. Accardi, Y. G. Lu, I. Volovich, "*Quantum theory and its stochastic limit, Ch 4.26*", 2002.
2. A. Barchielli, "*Continual measurements in Q.M. and quantum stochastic calculus, Sec 2.2*".

Nondemolition property

Lemma

$\{Z_t\}$ is a commutative process and $\mathcal{Z}_t := \mathrm{vN}\{Z_s | s \leq t\}$ is commutative.

Lemma : self-nondemolition property [1]

$\mathcal{Y}_t := \mathrm{vN}\{Y_s | 0 \leq s \leq t\}$ is commutative for all $t \in [0, T]$ with $T < \infty$.

Proof.

1. $E \in \mathcal{Z}_s$: an arbitrary projection in the range of the spectral measure of Z_s
2. Quantum Itô's formula on : $j_t(E) = U_t^* E U_t$
3. $j_t(E) = j_s(E) + \int_s^t j_r([L^*, E]) dA_r + \int_s^t j_r([E, L]) dA_r^\dagger + \int_s^t j_r(\mathcal{L}^*(E)) dr$
4. $E = \mathbb{1} \otimes E$ commutes with $L = L \otimes \mathbb{1}$ and $H = H \otimes \mathbb{1}$
5. $j_t(E) = j_s(E)$ for any $0 \leq s \leq t$
6. Bounded functional calculus $\Rightarrow \xi_{Y_s} = U_s^* \xi_{Z_s} U_s$ (ξ is the spectral measure)
7. $\mathcal{Z}_t$ commutative $\Rightarrow \mathcal{Y}_t$ commutative

Remark : $\{Y_t\}$ is commutative$\Rightarrow y(t) = \iota(Y_t)$ classical stochastic process

1. L. Bouten, R. van Handel, "*On the separation principle in quantum control, Prop 2.1* ", 2008.

Nondemolition property

Lemma : nondemolition property [1]

$j_t(X) \in \mathscr{Y}_t'$ for all $X \in \mathscr{B}(\mathcal{H}_S)$ and $t \in [0, T]$ with $T < \infty$.

Proof.

1. $\mathscr{Y}_t = U_t^* \mathscr{Z}_t U_t$
2. $j_t(X) = U_t^*(X \otimes \mathbb{1})U_t \in U_t^*(\mathscr{B}(\mathcal{H}_S) \otimes \mathbb{1})U_t$
3. $\mathscr{B}(\mathcal{H}_S) \otimes \mathbb{1}$ commutes with $\mathscr{Z}_t = \mathrm{vN}\{\mathbb{1} \otimes Z_s | 0 \leq s \leq t\}$
4. $j_t(X) \in \mathscr{Y}_t'$

Remark : Nondemolition property ensures the existence of conditional expectation.

1. L. Bouten, R. van Handel, "*On the separation principle in quantum control, Prop 2.1* ", 2008.

Conditional expectation and Bayes formula

Conditional expectation and Bayes formula

References :

1. R. van Handel, "*Filtering, stability, and robustness*", Ph.D. Thesis, 2007.
2. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering*", SIAM. J. Control Optim, 2007.
3. L. Bouten, R. van Handel, "*On the separation principle in quantum control*", Quantum Stoch. Inf. (pp. 206-238), 2008.
4. M. Takesaki, "*Conditional expectations in von Neumann algebras*", Journal of Functional Analysis, 1972.

Classical filtering theory [1]

Classical filtering theory on $(\Omega, \mathcal{F}, \mathbb{P})$: reference probability method

1. Conditional expectation : $\mathbb{E}(X|\mathcal{G})$ with $\mathcal{G} \subset \mathcal{F}$
2. Optimal estimation : $\mathbb{E}\big(X - \mathbb{E}(X|\mathcal{G})\big)^2 = \min_{Y \in L^2(\mathcal{G})} \mathbb{E}(X - Y)^2$.
3. Bayes formula : $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}^{\mathbb{Q}}(XM|\mathcal{G}) / \mathbb{E}^{\mathbb{Q}}(M|\mathcal{G})$ with $M = d\mathbb{P}/d\mathbb{Q}$
4. Girsanov theorem⇒define the reference probability $\mathbb{Q}_t$ s.t. $M_t = d\mathbb{P}/d\mathbb{Q}_t$
5. Kallianpur-Striebel formula $\mathbb{E}\big(f(x_t)|\mathcal{F}_t^y\big) = \mathbb{E}^{\mathbb{Q}_t}(M_t f(x_t)|\mathcal{F}_t^y) / \mathbb{E}^{\mathbb{Q}_t}(M_t|\mathcal{F}_t^y)$
6. Kushner-Stratonovich equation $\pi_t(f) := \mathbb{E}(f(x_t)|\mathcal{F}_t^y)$

1. J. Xiong, "*An introduction to stochastic filtering theory, Ch5*", Oxford, 2008

Quantum conditional expectation

Restrictions on quantum conditional expectation :

1. observations that is conditioned on must be commutative
   - It ensures that the ‘observations’ can be observed in a realization
   - It is guaranteed by self-nodemolition property.
2. conditioned observables must commute with (commutative) observations
   - It ensures that the joint statistic of events and observations is well defined
   - It is guaranteed by nodemolition property.

Definition : quantum conditional expectation

Let $(\mathcal{A}, \varphi)$ be a quantum probability space, $\mathcal{B} \subset \mathcal{A}$ be a commutative von Neumann subalgebra. Then $\varphi(\cdot|\mathcal{B}) : \mathcal{B}' \to \mathcal{B}$ is called (a version of) the conditional expectation given $\mathcal{B}$, if $\varphi\big(\varphi(X|\mathcal{B})S\big) = \varphi(XS)$ for all $X \in \mathcal{B}'$ and $S \in \mathcal{B}$.

Quantum conditional expectation

Lemma : existence and uniqueness of conditional expectation

For any commutative von Neumann algebra $\mathcal{B}$ and $X \in \mathcal{B}'$, the conditional expectation $\varphi(X|\mathcal{B})$ exists and is unique φ-a.s.

Proof.

- Existence
  - $X \in \mathcal{B}'$ is self-adjoint (classical conditional exp., spectral theorem)
  - $X \in \mathcal{B}'$ is not self-adjoint, $X = (X + X^*)/2 + i(X - X^*)/2i$
- Uniqueness
  - Let A and B be two versions of $\varphi(X|\mathcal{B}) \Rightarrow A - B \in \mathcal{B}$ and $A \in \mathcal{B}'$
  - $\varphi((A-B)^*(A-B)) = \|A-B\|_\varphi = 0.$

Quantum conditional expectation

Lemma : optimal estimation

For any $X \in \mathcal{B}'$, $\|X - \varphi(X|\mathcal{B})\|_\varphi = \min_{Y \in \mathcal{B}} \|X - Y\|_\varphi$.

Proof.

$$\begin{aligned}
\|X - Y\|_\varphi^2 &= \|X - \varphi(X|\mathcal{B}) + \varphi(X|\mathcal{B}) - Y\|_\varphi^2 \\
&= \|X - \varphi(X|\mathcal{B})\|_\varphi^2 + \|\varphi(X|\mathcal{B}) - Y\|_\varphi^2 \\
&\geq \|X - \varphi(X|\mathcal{B})\|_\varphi^2, \qquad \forall Y \in \mathcal{B}.
\end{aligned}$$

Quantum Bayes formula

Theorem : quantum Bayes formula

Let $(\mathcal{A}, \varphi)$ be a quantum probability space, $\mathcal{B} \subset \mathcal{A}$ be a commutative von Neumann subalgebra. Choose $V \in \mathcal{B}'$ s.t. $V^* V > 0$ and $\varphi(V^* V) = 1$. Then we can define a new state ϖ on $\mathcal{B}'$ by $\varpi(X) := \varphi(V^* XV)$, and

$$\varpi(X|\mathcal{B}) = \frac{\varphi(V^* XV|\mathcal{B})}{\varphi(V^* V|\mathcal{B})}, \quad \forall X \in \mathcal{B}'.$$

Proof.

$$\begin{aligned}
&\varphi(\varphi(V^* XV|\mathcal{B})K) = \varphi(V^* XVK) = \varphi(V^* XKV) = \varpi(XK) = \varpi(\varpi(X|\mathcal{B})K) \\
&= \varphi(V^* \varpi(X|\mathcal{B})KV) = \varphi(V^* V \varpi(X|\mathcal{B})K) = \varphi(\varphi(V^* V \varpi(X|\mathcal{B})K|\mathcal{B})) \\
&= \varphi(\varphi(V^* V|\mathcal{B})\varpi(X|\mathcal{B})K), \qquad \forall X \in \mathcal{B}', K \in \mathcal{B}.
\end{aligned}$$

Conditional expectation for unbounded observables

Definition : quantum conditional expectation

Let $(\mathcal{A}, \varphi)$ be a quantum probability space and $\mathcal{B} \subset \mathcal{A}$ be a commutative von Neumann subalgebra. Let $X \, \eta \, \mathcal{B}'$ be self-adjoint and suppose $\varphi(|X|) < \infty$. Then any self-adjoint operator $\varphi(X|\mathcal{B}) \, \eta \, \mathcal{B}$ satisfying $\varphi(\varphi(X|\mathcal{B}) \hat{\cdot} S) = \varphi(X \hat{\cdot} S)$ for all $S \in \mathcal{B}$ is called (a version of) the conditional expectation X given $\mathcal{B}$.

Remark :

- $\mathscr{S}(\mathrm{vN}(X, \mathcal{B}))$ forms a commutative $*$-algebra (with unit $\mathbb{1}$) under $\hat{+}$ and $\hat{\cdot}$. [1]
- Existence+uniqueness is ensured by classical condit. exp.+spectral thm

1. J. R. Ringrose, R. V. Kadison, "*Fundamentals of the theory of operator algebras, pp 351-356*", Vol 1, AMS, 1983.

Bayes formula (unbounded observables)

Motivation

Change-of-state operator V in Quantum Bayes formula may be **unbounded**

Settings for joint system :

- Hilbert space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_R$, $\dim(\mathcal{H}_S) = N < \infty$ and $\mathcal{H}_R$ is separable
- $\mathcal{A} = \mathscr{B}(\mathcal{H}_S) \otimes \mathscr{B}(\mathcal{H}_R)$, the normal state $\varphi = \varphi_S \otimes \varphi_R$
- Quantum probability space $(\mathcal{A}, \varphi)$
- $\mathcal{B}_R \subset \mathscr{B}(\mathcal{H}_R)$: a commutative von Neumann subalgebra
- $\mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R \subset (\mathbb{1} \otimes \mathcal{B}_R)'$

Bayes formula (unbounded observables)

1. $\mathscr{B}(\mathcal{H}_S) \simeq \mathscr{M}_N(\mathbb{C})$ with $\dim(\mathcal{H}_S) = N < \infty$
2. $\varphi_S(X_S) = \mathrm{Tr}(\rho X_S)$, $\forall X_S \in \mathscr{M}_N(\mathbb{C})$, where ρ is a density operator
3. $\mathcal{A} \simeq \mathscr{M}_N(\mathscr{B}(\mathcal{H}_R))$, the algebra of $N \times N$ matrices with $\mathscr{B}(\mathcal{H}_R)$-valued entries,

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1N} \\ \vdots & \ddots & \vdots \\ X_{N1} & \cdots & X_{NN} \end{bmatrix} \in \mathcal{A}, \quad X_{ij} \in \mathscr{B}(\mathcal{H}_R),$$

4. the state φ on $\mathcal{A}$ can be represented by

$$\varphi(X) = \mathrm{Tr}\left(\rho \begin{bmatrix} \varphi_R(X_{11}) & \cdots & \varphi_R(X_{1N}) \\ \vdots & \ddots & \vdots \\ \varphi_R(X_{N1}) & \cdots & \varphi_R(X_{NN}) \end{bmatrix}\right)$$

5. partial state of φ [1] : $\varphi_R(X) \in \mathscr{M}_N(\mathbb{C})$ and $\varphi_S(X) \in \mathscr{B}(\mathcal{H}_R)$, for all $X \in \mathcal{A}$
6. $\varphi(X) = \varphi_S(\varphi_R(X)) = \varphi_R(\varphi_S(X))$

1. J. Tomiyama, "*On the tensor products of von Neumann algebras*", Pacific J. Math., 1969.

Bayes formula (unbounded observables)

1. $\mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R \simeq \mathscr{M}_N(\mathcal{B}_R)$
2. $\mathscr{N}(\mathcal{B}_R)$: set of normal operators affiliated to $\mathcal{B}_R \Rightarrow *$-algebra under $\hat{+}$ and $\hat{\cdot}$
3. $\mathscr{M}_N(\mathscr{N}(\mathcal{B}_R))$ forms $*$-algebra under extension of $\hat{+}$ and $\hat{\cdot}$

Theorem : quantum Bayes formula [1]

Let $V \in \mathscr{M}_N(\mathscr{N}(\mathcal{B}_R))$ satisfying $\varphi(V^*V) = 1$. For $X \in \mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R$, define $\varpi(X) = \varphi(V^* \hat{\cdot} X \hat{\cdot} V)$. Then ϖ is a normal state on $\mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R$. Moreover, if $V^*V > 0$, then

$$\varpi(X|\mathbb{1} \otimes \mathcal{B}_R) = \mathbb{1} \otimes \frac{\varphi_S(V^* \hat{\cdot} X \hat{\cdot} V)}{\varphi_S(V^*V)}, \quad \forall X \in \mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R.$$

1. R. van Handel, "*Filtering, stability, and robustness, pp 107-108*", Ph.D. Thesis, 2007

Quantum filtering theory

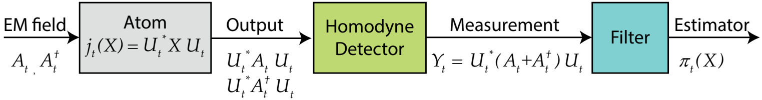
Quantum filtering theory

References :

1. R. van Handel, “*Filtering, stability, and robustness*”, Ph.D. Thesis, 2007.
2. L. Bouten, R. van Handel, M. James, “*An introduction to quantum filtering*”, SIAM. J. Control Optim, 2007.
3. L. Bouten, R. van Handel, “*On the separation principle in quantum control*”, Quantum Stoch. Inf. (pp. 206-238), 2008.
4. L. Accardi, Y. G. Lu, I. Volovich, “*Quantum theory and its stochastic limit, Ch 6*”, Springer, 2002.
5. V. P. Belavkin, “*Quantum stochastic calculus and quantum nonlinear filtering*”, Journal of Multivariate analysis, 1992.

Quantum filtering setup

- φ_R is state on $\mathscr{B}_{\mathbb{R}_+}$ is vacuum state $\langle e(0), \cdot\, e(0)\rangle$
- φ_S is state on $\mathscr{B}(\mathcal{H}_S)$, finite convex combination of $\langle v, \cdot\, v\rangle$ with $v \in \mathcal{H}_S$.
- $\mathcal{A} = \mathscr{B}(\mathcal{H}_S) \otimes \mathscr{B}(\Gamma_s(L^2(\mathbb{R}_+)))$ with $\dim(\mathcal{H}_S) = N < \infty$, $\varphi = \varphi_S \otimes \varphi_R$
- quantum probability space $(\mathcal{A}, \varphi)$



- self-nondemolition property$\Rightarrow$ Y_t can be observed in a realization
- nondemolition property$\Rightarrow$ $\pi_t(X) := \varphi(j_t(X)|\mathcal{Y}_t)$ is well defined

Objective of quantum filtering

Obtain an explicit expression of $\pi_t(X)$ in terms of the observations $\{Y_s\}_{0\le s\le t}$.

Change of state

Lemma

For all $X \in \mathscr{B}(\mathcal{H}_S) \otimes \mathbb{1}$, define $\varpi_t(X) = \varphi(U_t^* X U_t)$, then

$$\varphi(j_t(X)|\mathcal{Y}_t) = U_t^* \varpi_t(X|\mathcal{Z}_t) U_t, \quad \varphi - a.s.$$

Proof.

For any $S \in \mathcal{Z}_t$, then $U_t^* S U_t \in \mathcal{Y}_t = U_t^* \mathcal{Z}_t U_t$,

$$\varphi\left(U_t^* \varpi_t(X|\mathcal{Z}_t) U_t U_t^* S U_t\right) = \varpi_t\left(\varpi_t(X|\mathcal{Z}_t) S\right) = \varpi_t(XS) = \varphi(U_t^* X U_t U_t^* S U_t).$$

Remark :

- We should apply Bayes formula (bounded) for $\varpi_t(X|\mathcal{Z}_t)$
- $\varpi(X|\mathcal{Z}_t) = \varphi(U_t^* X U_t|\mathcal{Z}_t)/\varphi(U_t^* U_t|\mathcal{Z}_t)$ with $U_t \in \mathcal{Z}_t'$
- If $U_t \in \mathcal{Z}_t'$, $Y_t = U_t^* Z_t U_t = Z_t \Rightarrow$ system does not interact with field
- We cannot use U_t as change-of-state operator

Objective

Find $V_t \in \mathscr{M}_N(\mathscr{N}(\mathcal{Z}_t))$ s.t. $\varphi(U_t^* X U_t) = \varphi(V_t^* \hat{\cdot} X \hat{\cdot} V_t)$ for all $X \in \mathcal{B}(\mathcal{H}_S) \otimes \mathcal{Z}_t$

Change-of-state operator

Lemma

Let D_t, F_t, $\tilde{F}_t$ and G_t be bounded processes, and let

$$dV_t = \left(D_t dA_t^\dagger + F_t dA_t + G_t dt\right) V_t, \quad d\tilde{V}_t = \left(D_t dA_t^\dagger + \tilde{F}_t dA_t + G_t dt\right) \tilde{V}_t,$$

where $V_0 = \tilde{V}_0$. Then $V_t v \otimes e(0) = \tilde{V}_t v \otimes e(0)$ for all $v \in \mathcal{H}_S$.

Proof.

1. V_t and $\tilde{V}_t$ are uniquely defined admissible adapted processes
2. $\|(V_t - \tilde{V}_t) v \otimes e(0)\|^2 \leq C_T \int_0^t \|(V_s - \tilde{V}_s) v \otimes e(0)\|^2 ds$, with $t < T < \infty$
3. Gronwall inequality implies $\|(V_t - \tilde{V}_t) v \otimes e(0)\|^2 = 0$

Consequence :

$\varpi_t(X) = \varphi(U_t^* X U_t) = \langle U_t v \otimes e(0), X U_t v \otimes e(0)\rangle = \langle V_t v \otimes e(0), X V_t v \otimes e(0)\rangle$, where

$$dU_t = \left(L_t dA_t^\dagger - L_t^* dA_t - \tfrac{1}{2} L_t L_t^* dt - iH_t dt\right) U_t, \quad U_0 = \mathbb{1}$$

$$dV_t = \left(L_t (dA_t^\dagger + dA_t) - \tfrac{1}{2} L_t L_t^* dt - iH_t dt\right) V_t, \quad V_0 = \mathbb{1}$$

$dA_t e(0) = 0 \Rightarrow$ change of A_t-integral will not affect the impact of QSDE on vacuum

Change-of-state operator in vacuum state

Principal idea : [1 2 3]

1. $Z_t \overset{\text{spectral thm}}{\longleftrightarrow} z_t = \iota(Z_t)$ classical Brownian motion
2. $dV_t = \left(L_t(dA_t^\dagger + dA_t) - \frac{1}{2}L_tL_t^* dt - iH_t dt\right)V_t \overset{\text{spectral thm}}{\longleftrightarrow}$ matrix-valued Itô SDE
3. $V_t := \iota^{-1}(\text{solution of SDE}) \in \mathscr{M}_N(\mathscr{N}(\mathscr{Z}_t))$ is target change-of-state operator
4. new V_t and old V_t have same action on vacuum vector
5. $\left.\begin{array}{r}\text{spectral theorem}\\ V_t v \otimes e(0) = U_t v \otimes e(0)\end{array}\right\} \Rightarrow \varphi(U_t^* X U_t) = \varphi(V_t^* \hat{\,} X \hat{\,} V_t)$

1. R. van Handel, "*Filtering, stability, and robustness, pp 112-113* ", Ph.D. Thesis, 2007.
2. L. Accardi, Y. G. Lu, I. Volovich, "*Quantum theory and its stochastic limit, Ch 6.3*", 2002.
3. R. L. Hudson, K. R. Parthasarthy, "*Quantum Ito's formula and stochastic evolution, Sec 5* ".

Change-of-state operator in vacuum state

- $(\Omega, \mathcal{G}, \mu)$, ι : obtained by applying spectral thm on $\mathscr{Z}_t$
- $z_t = \iota(Z_t)$, $L_t = L_t \otimes \mathbb{1}$ and $H_t = H_t \otimes \mathbb{1}$ are bounded process

Lemma [1]

Let v_t be the solution of the matrix-valued Itô SDE,

$$dv_t = (L_t dz_t - \tfrac{1}{2} L_t L_t^* dt - iH_t dt) v_t, \qquad v_0 = \mathbb{1}.$$

Then V_t coincides with $\iota^{-1}(v_t) \in \mathscr{M}_N(\mathscr{N}(\mathscr{Z}_t))$ at least on a dense subdomain $\mathcal{H}_S \underline{\otimes} \mathcal{E}' \subset \mathcal{H}_S \underline{\otimes} \mathcal{E}$ which contains all vectors of the form $v \otimes e(0)$.

- Identify V_t with $\iota^{-1}(v_t)$

Lemma [1]

$\varpi_t(X) = \varphi(U_t^* X U_t) = \varphi(V_t^* \hat{}\, X \hat{}\, V_t)$ for all $X \in \mathcal{B}(\mathcal{H}_S) \otimes \mathscr{Z}_t$.

1. R. van Handel, "*Filtering, stability, and robustness, pp 112-113*", Ph.D. Thesis, 2007.

Quantum Kallianpur-Striebel formula (Bayes formula)

Theorem : quantum Bayes formula

Let $V \in \mathscr{M}_N(\mathscr{N}(\mathcal{B}_R))$ satisfying $\varphi(V^*V) = 1$. For $X \in \mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R$, define $\varpi(X) = \varphi(V^* \hat{\cdot} X \hat{\cdot} V)$. Then ϖ is a normal state on $\mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R$. Moreover, if $V^*V > 0$, then

$$\varpi(X|\mathbb{1} \otimes \mathcal{B}_R) = \mathbb{1} \otimes \frac{\varphi_S(V^* \hat{\cdot} X \hat{\cdot} V)}{\varphi_S(V^*V)}, \quad \forall X \in \mathscr{B}(\mathcal{H}_S) \otimes \mathcal{B}_R.$$

Theorem : quantum Kallianpur-Striebel formula

For all $X \in \mathscr{B}(\mathcal{H}_S)$, $\pi_t(X) = U_t^* \varpi_t(X \otimes \mathbb{1} | \mathcal{Z}_t) U_t$, where

$$\varpi_t(X \otimes \mathbb{1} | \mathcal{Z}_t) = \mathbb{1} \otimes \frac{\varphi_S(V_t^* \hat{\cdot} (X \otimes \mathbb{1}) \hat{\cdot} V_t)}{\varphi_S(V_t^* V_t)}.$$

- v_t is invertible [1] $\Rightarrow V_t^* V_t > 0$
- φ_S is finite convex combination of $\langle v, \cdot v \rangle$ with $v \in \mathcal{H}_S$.

1. P. E. Protter, "*Stochastic Integration and Differential Equations, pp 326* ", Springer, 2005.

Quantum Kallianpur-Striebel formula (Bayes formula)

- $(\Omega, \mathcal{F}, \mu)$, ι : obtained by applying spectral thm on $\mathscr{Y}_T = U_T^* \mathcal{Z}_T U_T$
- $\mathbb{P}$: probability measure on $\mathcal{F}$ induced by φ
- $y_t = \iota(Y_t)$, $L_t = L_t \otimes \mathbb{1}$ and $H_t = H_t \otimes \mathbb{1}$ are bounded process

Lemma [1]

Let $\bar{v}_t$ for $t \in [0, T]$ be the solution of the matrix-valued Itô SDE,

$$d\bar{v}_t = (L_t dy_t - \tfrac{1}{2} L_t L_t^* dt - iH_t dt)\bar{v}_t, \qquad \bar{v}_0 = \mathbb{1}.$$

For any $X \in \mathscr{B}(\mathcal{H}_S)$, define $\bar{\sigma}_t(X) := \varphi_S(\bar{v}_t^* X \bar{v}_t)$, $\bar{\pi}_t(X) := \bar{\sigma}_t(X)/\bar{\sigma}_t(\mathbb{1})$. Then,

- $\bar{\pi}_t(X) = \iota(\pi_t(X))$,
- $\{y_t\}_{t\in[0,T]}$ is a Brownian motion under $\mathbb{Q}$ s.t. $d\mathbb{P} = \bar{\sigma}_T(\mathbb{1}) d\mathbb{Q}$.

1. R. van Handel, "*Filtering, stability, and robustness, pp 112-113*", Ph.D. Thesis, 2007.

Input-output model for homodyne detection

Outline of proof.

- $\iota_Z : \mathscr{Z}_T \to L^\infty(\Omega_Z, \mathcal{F}_Z, \mu_Z)$, $\iota_Y : \mathscr{Y}_T \to L^\infty(\Omega_Y, \mathcal{F}_Y, \mu_Y)$
- $\iota : \mathscr{Y}_T \to L^\infty(\Omega_Y, \mathcal{F}_Y, \mu_Y)$ acts as $\iota(X) = \iota_Z(U_T X U_T^*)$
- probability measure on $\mathcal{F}_Z$: $\mathbb{E}^{\mathbb{P}}(\iota(X_y)) = \varphi(X_y)$, $\mathbb{E}^{\mathbb{Q}}(\iota_Z(X_z)) = \varphi(X_z)$
- $y_t = \iota(Y_t) = \iota_Z(Z_t) = z_t$: y_t is Brownian motion under $\mathbb{Q}$
- $\iota(\pi_t(X)) = \iota_Z(\varpi_t(X \otimes \mathbb{1} | \mathscr{Z}_t)) = \iota_Z\left(\mathbb{1} \otimes \frac{\varphi_S(V_t^* \hat{\cdot} (X \otimes \mathbb{1}) \hat{\cdot} V_t)}{\varphi_S(V_t^* V_t)}\right) = \bar{\pi}_t(X)$
- For any functional F_y of $\{y_t\}$,

$$\mathbb{E}^{\mathbb{P}}(F_y) = \varpi_T(\iota_Z^{-1}(F_y)) = \varphi(V_T^* \hat{\cdot} \iota_Z^{-1}(F_y) \hat{\cdot} V_T) = \mathbb{E}^{\mathbb{Q}}(\varphi_S(V_t^* V_t) F_y)$$

Innovations process

- $(\Omega, \mathcal{F}, \mu), \iota$: obtained by applying spectral thm on $\mathcal{Y}_T = U_T^* \mathcal{Z}_T U_T$
- $\mathbb{P}$: probability measure on $\mathcal{F}$ induced by φ
- $y_t = \iota(Y_t)$, $L_t = L_t \otimes \mathbb{1}$ is bounded process
- $\bar{\sigma}_t(X) := \varphi_S(\bar{v}_t^* X \bar{v}_t)$, $\bar{\pi}_t(X) := \bar{\sigma}_t(X)/\bar{\sigma}_t(\mathbb{1})$

Lemma

The innovations process $\hat{z}_t := y_t - \int_0^t \pi(L_t + L_t^*)ds$ is a Brownian motion under $\mathbb{P}$.

Proof.

1. Itô formula : $d\bar{\sigma}_t(\mathbb{1}) = \bar{\sigma}_t(L_t + L_t^*)dy_t = \bar{\pi}_t(L_t + L_t^*)\bar{\sigma}_t(\mathbb{1})dy_t$
2. $\bar{\sigma}_t(\mathbb{1})$ is Girsanov transformation

Quantum Kushner-Stratonovich equation

Theorem : quantum Kushner-Stratonovich equation

$\bar{\sigma}_t(X)$ and $\bar{\pi}_t(X)$ satisfy

$$d\bar{\sigma}_t(X) = \bar{\sigma}_t(\mathcal{L}_t^*(X))dt + \bar{\sigma}_t(L^*X + XL)dy_t,$$
$$d\bar{\pi}_t(X) = \bar{\pi}_t(\mathcal{L}_t^*(X))dt + \big(\bar{\pi}_t(L^*X + XL) - \bar{\pi}_t(L^* + L)\bar{\pi}_t(X)\big)d\bar{z}_t,$$

where $\bar{\sigma}_0(X) = \bar{\pi}_0(X) = \varphi_S(X)$ and $\mathcal{L}_t^*(X) := i[H_t, X] + L_t^*XL_t - \frac{1}{2}L_t^*L_tX - \frac{1}{2}XL_t^*L_t$.

Theorem : stochastic master equation

Let ρ_t be the random density matrix satisfying $\bar{\pi}_t(X) = \mathrm{Tr}(\rho_t X)$. Then ρ_t satisfies

$$d\rho_t = \mathcal{L}_t(\rho_t)dt + \big(L\rho_t + \rho_t L^* - \mathrm{Tr}((L_t + L_t^*)\rho_t)\rho_t\big)\big(dy_t - \mathrm{Tr}((L_t + L_t^*)\rho_t)dt\big),$$

where $\rho_0 = \rho$ and $\mathcal{L}_t(\rho) := i[\rho, H_t] + L_t\rho L_t^* - \frac{1}{2}L_t^*L_t\rho - \frac{1}{2}\rho L_t^*L_t$.

Theorem : quantum Zakai equation

Let ς_t be the random nonnegative self-adjoint matrix s.t. $\bar{\sigma}_t(X) = \mathrm{Tr}(\varsigma_t X)$. Then,

$$d\varsigma_t = \mathcal{L}_t(\rho_t)dt + (L\varsigma_t + \rho_t L^*)dy_t, \quad \varsigma_0 = \rho.$$

Moreover, $\rho_t = \varsigma_t / \mathrm{Tr}(\varsigma_t)$.

Quantum feedback control

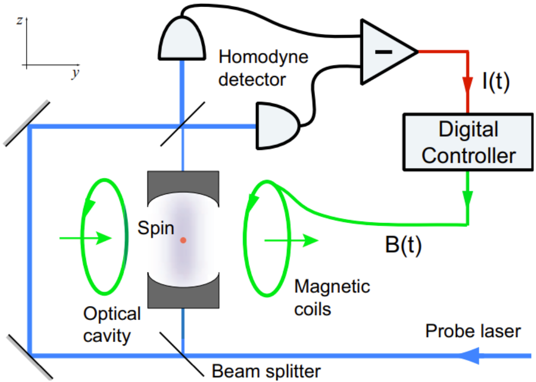


FIGURE – Experiment setup for feedback control of spin system, which interacts with an optical field measured continuously by homodyne detection. A magnetic field is used for the feedback [1].

1. R. van Handel, J.K. Stockton, H.Mabuchi, IEEE TAC, 2005.