Introduction to quantum feedback control

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Outline

Quantum stochastic calculus

Quantum stochastic calculus on restricted exponential domain

References :

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus* ", Birkhauser, 1992.
2. R. L. Hudson, "*An introduction to quantum stochastic calculus and some of its applications* ", Quant. Proba. Comm., 2003.
3. R. L. Hudson, K. R. Parthasarthy, "*Quantum Ito's formula and stochastic evolution* ", Comm. Math. Phys., 1984.
4. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering* ", SIAM. J. Control Optim, 2007.

Quantum probability space

Definition : quantum probability space

A quantum probability space (on $\mathcal{H}$) is a pair $(\mathcal{A}, \varphi)$, where

- $\mathcal{A}$ is a **von Neumann algebra** (on $\mathcal{H}$) ;
- φ is a **normal state** on $\mathcal{A}$.

Theorem : spectral theorem [1]

Let $(\mathcal{A}, \varphi)$ be a **commutative** quantum probability space on separable $\mathcal{H}$. Then there exist a finite measure space $(\Omega, \mathcal{F}, \mu)$, a $*$-isomorphism $\iota : \mathcal{A} \to L^\infty(\Omega, \mathcal{F}, \mu)$, and a probability measure $\mathbb{P} \ll \mu$ s.t. $\varphi(A) = \mathbb{E}^{\mathbb{P}}\big(\iota(A)\big)$ for all $A \in \mathcal{A}$.

Remark : commutative probability space is equivalent to classical probability space

1. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering, Thm 3.3*", 2007.

Symmetric Fock space and exponential vector

Definition : symmetric Fock space

A symmetric (or bosonic) Fock space over $\mathcal{H}$ is $\Gamma_s(\mathcal{H}) := \mathbb{C} \oplus \bigoplus_{n=1}^{+\infty} \mathcal{H}^{\circ n}$,
$\mathcal{H}$ is called single-particle Hilbert space.

- **exponential vector** : $e(u) = \bigoplus_{n=0}^{+\infty} \frac{u^{\otimes n}}{\sqrt{n!}} \in \Gamma_s(\mathcal{H})$ with $u \in \mathcal{H}$
- **vacuum vector** : $e(0) = 1 \oplus 0 \oplus 0 \oplus \dots$
- **exponential domain** : $\mathcal{E}(\mathcal{H}) := \mathrm{span}\{e(u) | \, u \in \mathcal{H}\}$ is **dense** in $\Gamma_s(\mathcal{H})$

Fundamental stochastic processes

Theorem : Weyl operator, field operator, differential second quantization

- $\forall u, v \in \mathcal{H}$, $\forall$ unitary operator U on $\mathcal{H}$, $\exists!$ unitary operator (**Weyl operator**) $W(u,U)$ on $\Gamma_s(\mathcal{H})$ s.t. $W(u,U)e(v) = e^{-\langle u, Uv\rangle - \|u\|^2/2} e(Uv+u)$.
- $W_u := W(u, \mathbb{1}) \Rightarrow B(u)$ s.t. $W_{tu} = e^{itB(u)}$ (Stone's theorem)
- $\Gamma(U) := W(0,U) \Rightarrow \Lambda(A)$ s.t. $\Gamma(e^{itA}) = e^{it\Lambda(A)}$ (Stone's theorem)

Definition : fundamental stochastic processes on $\Gamma_s(L^2(\mathbb{R}_+))$

- quadratures : $Q_t := B(i\mathbb{1}_{[0,t]})$, $P_t := B(-\mathbb{1}_{[0,t]})$
- **gauge process** : $\Lambda_t := \Lambda(M_{\mathbb{1}_{[0,t]}})$ with $M_{\mathbb{1}_{[0,t]}} f = \mathbb{1}_{[0,t]} f$, for $f \in L^2(\mathbb{R}_+)$

* Q_t, P_t, Λ_t are commutative, they do not commute with each other ;
* $\imath(Q_t), \imath(P_t)$ are **Wiener** processes in $\langle e(0), \cdot\, e(0)\rangle$;
* $\imath(\Lambda_t)$ is **Poisson** process with intensity $|f(t)|^2$ in $e^{-\|f\|^2}\langle e(f), \cdot\, e(f)\rangle$.

Quantum stochastic calculus

Motivation

- interaction between system and field $\Rightarrow$ quantum stochastic integral
- domain problem $\Rightarrow$ restricted exponential domain (allow strong limits)

Definition : quantum probability space

- $\mathcal{H}_s$: Hilbert space of the initial system, φ_s : state on $\mathscr{B}(\mathcal{H}_s)$
- $\Gamma_s(L^2(\mathbb{R}_+))$: Hilbert space of the field, φ_f : state on $\mathscr{B}(\Gamma_s(L^2(\mathbb{R}_+)))$
- $\mathcal{A} := \mathscr{B}(\mathcal{H}_s) \otimes \mathscr{B}(\Gamma_s(L^2(\mathbb{R}_+)))$, $\varphi := \varphi_s \otimes \varphi_f$
- quantum probability space : $(\mathcal{A}, \varphi)$

Remark : suppose $\mathcal{H}_s$ is finite-dimensional.

Exponential property

- **exponential vector** : $e(u) = \bigoplus_{n=0}^{+\infty} u^{\otimes n}/\sqrt{n!} \in \Gamma_s(\mathcal{H})$ with $u \in \mathcal{H}$
- **vacuum vector** : $e(0) = 1 \oplus 0 \oplus 0 \oplus \dots$
- **exponential domain** : $\mathcal{E}(\mathcal{H}) := \mathrm{span}\{e(u) | u \in \mathcal{H}\}$ **dense** [1] in $\Gamma_s(\mathcal{H})$

Theorem : Exponential property [1]

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then there exists a *unique unitary isomorphism* $U : \Gamma_s(\mathcal{H}_1 \oplus \mathcal{H}_2) \to \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ s.t. $Ue(u \oplus v) = e(u) \otimes e(v)$.

Tensor product splitting of $e(f)$:

- $f \in L^2(\mathbb{R}_+)$, $f_{s]} := f\mathbb{1}_{[0,s]}$, $f_{[s,t]} := f\mathbb{1}_{[s,t]}$, $f_{[t} := f\mathbb{1}_{[t,\infty)}$
- $e(f) \in \Gamma_s\big(L^2(\mathbb{R}_+)\big)$ satisfies $e(f) = e(f_{s]}) \otimes e(f_{[s,t]}) \otimes e(f_{[t})$ (through U)

Tensor product splitting of $\Gamma_s\big(L^2(\mathbb{R}_+)\big)$:

- $L^2(\mathbb{R}_+) = L^2[0,s] \oplus L^2[s,t] \oplus L^2[t,\infty)$
- $\Gamma_s\big(L^2(\mathbb{R}_+)\big) = \Gamma_s(L^2[0,s]) \otimes \Gamma_s(L^2[s,t]) \otimes \Gamma_s(L^2[t,\infty))$ (through U)

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, pp 126-127*", 1992.

Calculus on restricted exponential domain [1]

Definition : restricted exponential domain

$\mathcal{D} := \mathrm{span}\{e(f) | f \in L^2(\mathbb{R}_+) \cap L^{\infty,\mathrm{loc}}(\mathbb{R}_+)\} \subset \mathcal{E}(L^2(\mathbb{R}_+))$ is **dense** in $\Gamma_s(L^2(\mathbb{R}_+))$.

Remark : $L^{\infty,\mathrm{loc}}$ prevents Λ_t from blowing up at finite time.

Definition : allowable operator

- $\mathcal{H}_s \underline{\otimes} \mathcal{D} := \mathrm{span}\{f \otimes e(u) | f \in \mathcal{H}_s, u \in L^2 \cap L^{\infty,\mathrm{loc}}\}$
- X on $\mathcal{H}_s \otimes \Gamma_s$ is **allowable**, if $D(X) = \mathcal{H}_s \underline{\otimes} \mathcal{D}$ and $D(X^*) \supset D(X)$.
- For an allowable X, $X^\dagger$ is the restriction of X^* to $\mathcal{H}_s \underline{\otimes} \mathcal{D}$.

Definition : admissible process

- An **admissible** process is a family of $\{X_t\}_{t \in \mathbb{R}_+}$ of allowable operators.
- An admissible process X_t is **adapted**, if $X_t \psi \otimes e(f) = (X_{t]} \psi \otimes e(f_{t]})) \otimes e(f_{[t})$.

1. R. L. Hudson, "*An introduction to quantum stochastic calculus and some of its applications*", Quant. Proba. Comm., 2003.

Quantum fundamental noises

$$Q_t = B(i\mathbb{1}_{[0,t]}),\ P_t = B(-\mathbb{1}_{[0,t]}),\ \Lambda_t = \Lambda(M_{\mathbb{1}_{[0,t]}})$$

Definition : quantum fundamental noises

Three **admissible adapted** quantum fundamental noises are defined as

- annihilation process : $A_t = \mathbb{1} \otimes (Q_t + iP_t)/2$,
- creation process : $A_t^\dagger = \mathbb{1} \otimes (Q_t - iP_t)/2$, $(A_t)^\dagger = A_t^\dagger$
- gauge process : $\Lambda_t = \mathbb{1} \otimes \Lambda_t$

Properties [1]

1. $W(u_1 \oplus u_2, U_1 \oplus U_2) = W(u_1, U_1) \otimes W(u_2, U_2)$
2. $B(u_1 \oplus u_2) e(v_1 \oplus v_2) = [B(u_1) e(v_1)] \otimes e(v_2) + e(v_1) \otimes [B(u_2) e(v_2)]$
3. $\Lambda(A_1 \oplus A_2) e(v_1 \oplus v_2) = [\Lambda(A_1) e(v_1)] \otimes e(v_2) + e(v_1) \otimes [\Lambda(A_1) e(v_2)]$

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, pp 150*", 1992.

Simple quantum stochastic integral

Lemma

Let $M_t \in \{A_t, A_t^\dagger, \Lambda_t\}$. For $0 \leq s < t < \infty$, $\exists$ an operator $\Delta M_{[s,t]}$ on $\mathcal{D}_{[s,t]}$ s.t.

$$(M_t - M_s)\psi \otimes e(u) = \psi \otimes e(u_{s]}) \otimes \Delta M_{[s,t]} e(u_{[s,t]}) \otimes e(u_{[t}).$$

Let L_s be admissible and adapted ($L_s = L_{s]} \otimes \mathbb{1}$), $L_s(M_t - M_s) = L_{s]} \otimes \Delta M_{[s,t]} \otimes \mathbb{1}$.

Proof. M_t is adapted, then apply the above property.

Definition : simple quantum stochastic integral

- An admissible process L_t is **simple**, if $\exists$ a sequence $0 = t_0 < \cdots < t_n < \cdots$ where $t_n \to \infty$ as $n \to \infty$ s.t. $L_t = \sum_{n=0}^{\infty} L_{t_n} \mathbb{1}_{[t_n, t_{n+1})}(t)$.
- Let simple process L_t be adapted, simple quantum stochastic integral on $\mathcal{H}_s \underline{\otimes} \mathcal{D}$ is defined as $\int_0^T L_t dM_t := \sum_{n=0}^{\infty} L_{t_n}(M_{t_{n+1}\wedge T} - M_{t_n \wedge T})$.

Quantum stochastic integral [1]

Lemma : fundamental estimate

For any simple adapted process L_t, $\| \int_0^T L_t dM_t f \otimes e(u)\|^2 \leq C_T^u \int_0^T \|L_t f \otimes e(u)\|^2 dt$.

Definition : stochastically-integrable

An adapted X_t is **stochastically-integrable** (S.I.), if $\exists$ a sequence of simple adapted processes $X_t^{(n)}$ s.t. $\lim_{n\to\infty} \int_0^T \|(X_t - X_t^{(n)}) f \otimes e(u)\|^2 dt = 0$.

Lemma

Suppose that an adapted process X_t satisfies : for each $f \otimes e(u) \in \mathcal{H}_s \underline{\otimes} \mathcal{D}$

- the mapping $t \to X_t f \otimes e(u)$ is left continuous ;
- $\sup_{0\leq s\leq t} \|X_t f \otimes e(u)\| < \infty$ for each t.

Then X_t is **stochastically-integrable**.

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, pp 188-190* ", 1992.

Quantum stochastic integral

1. X_t is stochastically-integrable
2. $\left\| \left(\int_0^T X_t^{(n)} dM_t - \int_0^T X_t^{(m)} dM_t \right) f \otimes e(u) \right\|^2 \leq C_T^u \int_0^T \|(X_t^{(n)} - X_t^{(m)}) f \otimes e(u)\|^2 dt$
3. $\int_0^T X_t^{(n)} dM_t f \otimes e(u)$ converges in $\mathcal{H}_s \otimes \Gamma_s$, for all $f \otimes e(u) \in \mathcal{H}_s \underline{\otimes} \mathcal{D}$

Definition : quantum stochastic integral

For any **stochastically-integrable** process X_t, the quantum stochastic integral is defined by actions on $\mathcal{H}_s \underline{\otimes} \mathcal{D}$: $\int_0^T X_t dM_t f \otimes e(u) := \lim_{n \to \infty} \int_0^T X_t^{(n)} dM_t f \otimes e(u)$, the limit does not depend on the choice of $X^{(n)}$.

For **stochastically-integrable** (**S.I.**) processes (E_t, F_t, G_t, H_t), define

$$I_T := \int_0^T (E_t d\Lambda_t + F_t dA_t + G_t dA_t^\dagger + H_t dt)$$

First fundamental lemma

- $I_T := \int_0^T (E_t d\Lambda_t + F_t dA_t + G_t dA_t^\dagger + H_t dt)$

Theorem : first fundamental lemma [1]

Let (E_t, F_t, G_t, H_t) be stochastically-integrable. For all $f \otimes e(u), g \otimes e(v) \in \mathcal{H}_S \underline{\otimes} \mathcal{D}$,

$$\begin{aligned}
&\langle f \otimes e(u), I_T\, g \otimes e(v) \rangle \\
&= \int_0^T \langle f \otimes e(u), (u(t)^* E_t v(t) + F_t v(t) + u^*(t) G_t + H_t) g \otimes e(v) \rangle \, dt \\
&= \int_0^T \left\langle f \otimes e(u), \begin{bmatrix} 1, u^*(t) \end{bmatrix} \begin{bmatrix} H_t & F_t \\ G_t & E_t \end{bmatrix} \begin{bmatrix} 1 \\ v(t) \end{bmatrix} g \otimes e(v) \right\rangle dt.
\end{aligned}$$

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, Prop 25.9*", 1992.

First fundamental lemma

Outline of proof.

1. First fundamental lemma for simple adapted process [1] :

$$\Big\langle f \otimes e(u), \int_0^T E_t^{(n)} d\Lambda_t\, g \otimes e(v) \Big\rangle = \int_0^T u(t)^* v(t) \langle f \otimes e(u), E_t^{(n)} g \otimes e(v) \rangle dt.$$

2. Let S.I. processes be approximated by a sequence of simple processes :

$$\Big|\Big\langle f \otimes e(u), \int_0^T E_t d\Lambda_t\, g \otimes e(v) \Big\rangle - \int_0^T u(t)^* v(t) \langle f \otimes e(u), E_t\, g \otimes e(v) \rangle dt \Big|$$

$$= \lim_{n\to\infty} \Big|\Big\langle f \otimes e(u), \int_0^T E_t^{(n)} d\Lambda_t\, g \otimes e(v) \Big\rangle - \int_0^T u^* v \langle f \otimes e(u), E_t\, g \otimes e(v) \rangle dt \Big|$$

$$= \lim_{n\to\infty} \Big| \int_0^T u^* v \langle f \otimes e(u), (E_t^{(n)} - E_t)\, g \otimes e(v) \rangle dt \Big|$$

$$\leq \lim_{n\to\infty} C_T \int_0^T \|f \otimes e(u)\| \|(E_t^{(n)} - E_t)\, g \otimes e(v)\| dt, \quad C_T := \sup\{|u^* v| : t \in [0, T]\}$$

$$\leq \lim_{n\to\infty} C_T \sqrt{T} \|f \otimes e(u)\| \Big(\int_0^T \|(E_t^{(n)} - E_t)\, g \otimes e(v)\|^2 dt \Big)^{1/2} = 0.$$

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1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, Prop 25.1* ", 1992.

Second fundamental lemma

Theorem : second fundamental lemma [1]

Let (E_t, F_t, G_t, H_t), (E'_t, F'_t, G'_t, H'_t) be S.I. For all $f \otimes e(u), g \otimes e(v) \in \mathcal{H}_S \underline{\otimes} \mathcal{D}$,

$$\begin{aligned}
&\langle I'_T f \otimes e(u), I_T g \otimes e(v) \rangle \\
&= \int_0^T \big\langle I'_T f \otimes e(u), \big(u(t)^* E_t v(t) + F_t v(t) + u(t)^* G_t + H_t\big) g \otimes e(v) \big\rangle \, dt \\
&\quad + \int_0^T \big\langle \big(v(t)^* E'_t u(t) + F'_t u(t) + v^*(t) G'_t + H'_t\big) f \otimes e(u), I_T g \otimes e(v) \big\rangle \, dt \\
&\quad + \int_0^T \big\langle (E'_t u(t) + G'_t) f \otimes e(u), (E_t v(t) + G_t) g \otimes e(v) \big\rangle \, dt.
\end{aligned}$$

Lemma [1]

Let (E_t, F_t, G_t, H_t) be stochastically-integrable, then $(E_t^\dagger, F_t^\dagger, G_t^\dagger, H_t^\dagger)$ are S.I. and $I_T^\dagger = \int_0^T (E_t^\dagger d\Lambda_t + G_t^\dagger dA_t + F_t^\dagger dA_t^\dagger + H_t^\dagger dt)$.

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, pp 191-192*", 1992.

Quantum Ito formula

$$I_T = \int_0^T (E_t d\Lambda_t + F_t dA_t + G_t dA_t^\dagger + H_t dt)$$

$$I'_T = \int_0^T (E'_t d\Lambda_t + F'_t dA_t + G'_t dA_t^\dagger + H'_t dt)$$

$$\langle f \otimes e(u), I_T g \otimes e(v) \rangle$$
$$= \int_0^T \left\langle f \otimes e(u), \left[1, u^*(t)\right] \begin{bmatrix} H_t & F_t \\ G_t & E_t \end{bmatrix} \begin{bmatrix} 1 \\ v(t) \end{bmatrix} g \otimes e(v) \right\rangle dt$$

$$\langle (I'_T)^\dagger f \otimes e(u), I_T g \otimes e(v) \rangle = \langle f \otimes e(u), I'_T I_T g \otimes e(v) \rangle$$
$$= \int_0^T \left\langle f \otimes e(u), \left[1, u^*(t)\right] \begin{bmatrix} I'_t H_t + H'_t I_t + F'_t G_t & I'_t F_t + F'_t I_t + F'_t E_t \\ I'_t G_t + G'_t I_t + G'_t I_t & I'_t E_t + E'_t I_t + E'_t E_t \end{bmatrix} \begin{bmatrix} 1 \\ v(t) \end{bmatrix} g \otimes e(v) \right\rangle dt$$

Quantum Ito formula

Theorem : quantum Itô rule [1]

Let (E'_t, F'_t, G'_t, H'_t) and $((E'_t)^\dagger, (F'_t)^\dagger, (G'_t)^\dagger, (H'_t)^\dagger)$ be S.I. pair, and (E_t, F_t, G_t, H_t) be S.I. Let I_t and I'_t be quantum stochastic integrals of the form

$$dI_t = E_t d\Lambda_t + F_t dA_t + G_t dA_t^\dagger + H_t dt, \quad dI'_t = E'_t d\Lambda_t + F'_t dA_t + G'_t dA_t^\dagger + H'_t dt,$$

Suppose that $I'_t I_t$ is adapted process and $I'_t E_t$, $I'_t F_t$, $I'_t G_t$, $I'_t H_t$, $E'_t I_t$, $F'_t I_t$, $G'_t I_t$, $H'_t I_t$, $E'_t E_t$, $F'_t E_t$, $E'_t G_t$, $F'_t G_t$ are S.I. Then $d(I'_t I_t) = I'_t dI_t + (dI'_t) I_t + dI'_t dI_t$, where

$$I'_t dI_t = I'_t E_t d\Lambda_t + I'_t F_t dA_t + I'_t G_t dA_t^\dagger + I'_t H_t dt,$$

$$(dI'_t) I_t = E'_t I_t d\Lambda_t + F'_t I_t dA_t + G'_t I_t dA_t^\dagger + H'_t I_t dt,$$

$$dI'_t dI_t = E'_t E_t d\Lambda_t + F'_t E_t dA_t + E'_t G_t dA_t^\dagger + F'_t G_t dt,$$

are evaluated according to the following rules

$dI'_t \setminus dI_t$	dA_t	$d\Lambda_t$	$dA_t^\dagger$	dt
dA_t	0	dA_t	dt	0
$d\Lambda_t$	0	$d\Lambda_t$	$dA_t^\dagger$	0
$dA_t^\dagger$	0	0	0	0
dt	0	0	0	0

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1. K. R. Parthasarthy, “*An introduction to quantum stochastic calculus, Prop 25.26*”, 1992.

Quantum stochastic differential equation

Quantum stochastic differential equation

References :

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus* ", Birkhauser, 1992.
2. R. L. Hudson, K. R. Parthasarthy, "*Quantum Ito's formula and stochastic evolution* ", Comm. Math. Phys., 1984.
3. P.Meyer, "*Quantum probability for probabilists* ", LNM (Vol 1538), Springer, 1993.
4. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering* ", SIAM. J. Control Optim, 2007.

Quantum stochastic differential equation

Motivation and principal idea

1. Quantum stochastic integral is symmetric on $\mathcal{H}_s \underline{\otimes} \mathcal{D}$ (dense in $\mathcal{H}_s \otimes \Gamma_s$)
2. Symmetric operators may have many (even no) self-adjoint extensions [1]
3. Cannot interpret quantum stochastic integrals as observables
4. Conditions ensure quantum stochastic integral is bounded on $\mathcal{H}_s \underline{\otimes} \mathcal{D}$
5. Extend to a unique bounded operator on $\mathcal{H}_s \otimes \Gamma_s(L^2(\mathbb{R}_+))$ [1]

Remark : $\mathcal{H}_s$ is finite dimensional.

1. M. Reed, B. Simon, "*Method of modern mathematical physics, pp 257-259*", 1972.
2. M. Reed, B. Simon, "*Method of modern mathematical physics, Thm I.7*", 1972.

Quantum stochastic differential equation

Theorem : existence and uniqueness

Let $\mathcal{H}_s$ be finite dimensional. Consider a QSDE

$$V_t = \mathbb{1} + \int_0^t \left(L_1(s) d\Lambda_s + L_2(s) dA_s + L_3(s) dA_s^\dagger + L_4(s) ds \right) Vs,$$

where $L_n(t)$ is bounded process of the form $L_n(t) \otimes \mathbb{1}$ on $\mathcal{H}_s \otimes \Gamma_s$ for $n = 1, 2, 3, 4$. Then, there exists a unique admissible adapted process V_t solving the QSDE.

Remark :

- Convention : $dM_s V_s = V_s dM_s$, $ds V_s = V_s d_s$.
- The proof can be showed by Picard iteration [1].
- QSDE with bounded (unbounded) coefficients on infinite-dimensional $\mathcal{H}_s$ and Fock space with infinite multiplicity is discussed in [2,3].

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1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, Prop 26.1* ", 1992.
2. P. Meyer, "*Quantum probability for probabilists, Ch VI.4* ", LNM1538, 1993.
3. F. Fagnola, "*QSDEs and dilation of completely positive semigroups*", LNM1881, 2006.

Hudson-Parthasarathy equation

Theorem : unitary solution [1]

The unique solution of QSDE $dV_t = \big(L_1(t)d\Lambda_t + L_2(t)dA_t + L_3(t)dA_t^\dagger + L_4(t)dt\big)Vt$, with $V_0 = \mathbb{1}$ is **unitary** if and only if

$$(L_1(t), L_2(t), L_3(t), L_4(t)) = (W_t - \mathbb{1}, -L_t^* W_t, L_t, \tfrac{1}{2}L_t L_t^* - iH_t),$$

where W_t, L_t, H_t are bounded processes of the form $X_t \otimes \mathbb{1}$ on $\mathcal{H}_s \otimes \Gamma_s$, W_t is unitary and H_t is self-adjoint.

Hudson-Parthasarathy (H-P) equation :

$$dU_t = \big((W_t - \mathbb{1})d\Lambda_t - L_t^* W_t dA_t + L_t dA_t^\dagger + \tfrac{1}{2}L_t L_t^* dt - iH_t dt\big)U_t, \quad U_0 = \mathbb{1}.$$

Remark :

- U_t is unitary adapted process on $\mathcal{H}_s \otimes \Gamma_s(L^2(\mathbb{R}_+))$, not a group.
- $V_t := \begin{cases} \Theta_t U_t & \text{if } t \geq 0 \\ U_{|t|}\Theta_t & \text{if } t \leq 0 \end{cases}$ strongly continuous group of unitary operators on $\mathcal{H}_s \otimes \Gamma_s(L^2(\mathbb{R}))$, with $\Theta_t e(f) = e(\theta_t f)$ and $(\theta_t f)(r) = f(r+t)$, $\forall f \in L^2(\mathbb{R})$. [2]

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, Cor 26.4*", 1992.
1. M. Gregoratti, "*On the Hamiltonian operator associated to some QSDEs*", 2000.

Quantum stochastic flow

1. H-P eq : $dU_t = \big((W_t - \mathbb{1})d\Lambda_t - L_t^* W_t dA_t + L_t dA_t^\dagger + \frac{1}{2}L_t L_t^* dt - iH_t dt\big)U_t$
2. $W_t = \mathbb{1}$, $L_t = 0 \Rightarrow dU_t = iH_t U_t dt$, Schrödinger eq with Hamiltonian H_t
3. H-P equation can be interpreted as noisy Schrödinger equation
4. observable X on $\mathcal{H}_s \Rightarrow$ define quantum stochastic flow $j_t(X) := U_t^*(X \otimes \mathbb{1})U_t$

Remark : (H.P.) $j_t(X) = V_t^*(X \otimes \mathbb{1})V_t = U_t^*(X \otimes \mathbb{1})U_t$. ($X \otimes \mathbb{1}$ commutes with Θ_t)

Lemma : quantum stochastic flow [1]

For any X on $\mathcal{H}_s$, $\{j_t(X)\}_{t \in \mathbb{R}_+}$ is an adapted process satisfying

$$dj_t(X) = j_t(W_t^* X W_t - X)d\Lambda_t + j_t([L_t^*, X]W_t)dA_t + j_t(W_t^*[X, L_t])dA_t^\dagger + j_t(\mathcal{L}^*(X))dt,$$

with $\mathcal{L}^*(X) := i[H_t, X] + L_t X L_t^* - \frac{1}{2}(L_t L_t^* X + X L_t L_t^*)$ and $j_0(X) = X \otimes \mathbb{1}$. Then,

- $X \to j_t(X)$ is $*$-homomorphism from $\mathscr{B}(H_s)$ to $\{X \otimes \mathbb{1}_{[t} | X \in \mathscr{B}(\mathcal{H}_s \otimes \Gamma_s[0,t])\}$.
- For any X on $\mathcal{H}_s$, $t \to j_t(X)$ is strongly continuous.

1. K. R. Parthasarthy, “*An introduction to quantum stochastic calculus, Cor 26.5*”, 1992.

From quantum stochastic flow to master equation

1. quantum probability space : $(\mathscr{B}(\mathcal{H}_s) \otimes \mathscr{B}(\Gamma_s(\mathbb{R}_+)), \varphi_s \otimes \varphi_f)$
2. $\varphi_s = \mathrm{Tr}(\rho \cdot)$ with density matrix ρ on $\mathcal{H}_s$, $\varphi_f = \langle e(0), \cdot\, e(0) \rangle$ is vacuum state
3. define $T_t : \mathscr{B}(\mathcal{H}_s) \to \mathscr{B}(\mathcal{H}_s)$ by averaging $j_t(X)$ under vacuum state,
$$\langle f, T_t(X) g \rangle = \langle f \otimes e(0), j_t(X) g \otimes e(0) \rangle, \quad \forall f, g \in \mathcal{H}_s$$
4. $T_t = e^{t\mathcal{L}^*}$ is one parameter semigroup of completely positive linear operators in $\mathscr{B}(\mathcal{H}_s)$ generated by the adjoint Lindblad operator $\mathcal{L}^*$ [1]
$$\mathcal{L}^*(X) := i[H_t, X] + L_t X L_t^* - \frac{1}{2}(L_t L_t^* X + X L_t L_t^*)$$
5. $\mathrm{Tr}(\rho T_t(X)) = \mathrm{Tr}(\rho_t X)$, where $\rho_t = P_t(\rho)$, one parameter semigroup generated by Lindblad operator $\mathcal{L}$.

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, pp 214–216* ", 1992.

Input-output model for homodyne detection

Emergence of H-P equations in physical applications

1. (finite-dimensional) main system interacts with an electromagnetic field
2. dynamics of the whole system can be described by
$$\frac{d}{dt}\widetilde{U}(t) = \big(-iH_t + L_t\tilde{a}^*(t,0) - L_t^*\tilde{a}(t,0)\big)\widetilde{U}(t), \quad \widetilde{U}(0) = \mathbb{1},$$
3. weak coupling limit [1] $\Rightarrow$ H-P equation
$$dU_t = \big(L_t^* dA_t + L_t dA_t^\dagger + \tfrac{1}{2}L_t L_t^* dt - iH_t dt\big)U_t, \quad U_0 = \mathbb{1}.$$

Evolution of system :

- $dj_t(X) = j_t(\mathcal{L}^*(X))dt + j_t([X, L_t])dA_t^\dagger + j_t([L_t^*, X])dA_t$

Evolution of field : influence of system on field

- observation process : $Y_t := U_t^* Z_t U_t$ with $Z_t := A_t + A_t^\dagger$ for homodyne detection
- Z_t corresponds to input field
- Y_t corresponds to output field (after interaction of input-field and system)

1. L. Accardi, A. Frigerio, Y. Lu, "*The weak coupling limit as a quantum functional central limit*".

Input-output model for homodyne detection

System-observation pair (homodyne detection) :

$$dj_t(X) = j_t(\mathcal{L}^*(X))dt + j_t([L^*, X])dA_t + j_t([X, L])dA_t^\dagger, \quad j_0(X) = X \otimes \mathbb{1},$$
$$dY_t = j_t(L + L^*)dt + dA_t + dA_t^\dagger,$$

where

- $\mathcal{L}^*(X) := i[H_t, X] + LXL^* - \frac{1}{2}(LL^*X + XLL^*)$
- X and L are constants on $\mathcal{H}_s$ in the form $X = X \otimes \mathbb{1}$
- $H_t = H_0 + \sum_j u_j(t) H_j$, H_j are control Hamiltonians, bounded $u_j(t) \in \mathbb{R}$ relate to amplitude of interaction between system and control fields [1].

Problem on observation : $Y_t = U_t^*(A_t + A_t^\dagger)U_t$

Does this observation process make sense ? Can we observe Y_t in laboratory ?

- $A_t + A_t^\dagger$ is essentially self-adjoint on $\mathcal{E}$ (Interaction picture)
- Self-nondemolition property

1. H. M. Wiseman, G. J. Milburn, "*Quantum measurement and control, Ch 3* ", 2009.

Quantum feedback control

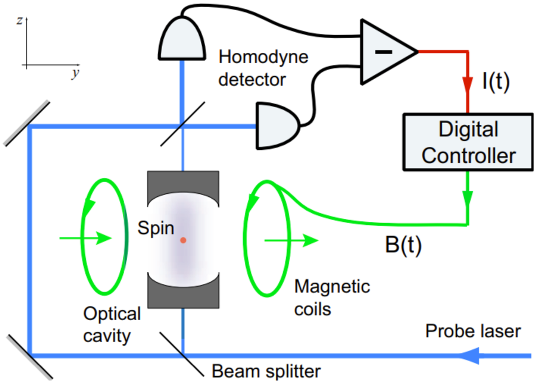

FIGURE – Experiment setup for feedback control of spin system, which interacts with an optical field measured continuously by homodyne detection. A magnetic field is used for the feedback [1].

1. R. van Handel, J.K. Stockton, H.Mabuchi, IEEE TAC, 2005.