Introduction to quantum feedback control

CY-McGill Mathematical Physics Weekly Seminar

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Outline

1 Quantum probability theory

2 Quantum stochastic process : Wiener and Poisson processes

Quantum probability theory

Quantum probability theory

References :

1. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering*", SIAM. J. Control Optim, 2007.
2. R. van Handel, "*Filtering, stability, and robustness*", Ph.D. Thesis, 2007.
3. L. Bouten, "*Applications of quantum stochastic processes in quantum optics*", Quantum Potential Theory, pp 277-307, Springer, 2008.
4. P.Meyer, "*Quantum probability for probabilists*", LNM (Vol 1538), Springer, 1993.
5. R. V. Kadison, J. R. Ringrose "*Funfamentals of the theory of operator algebras*", Vol 1, AMS, 1983.
6. M. Reed, B. Simon, "*Method of modern mathematical physics*", Academic press, 1972.

Quantum probability theory at Hilbert space level [1]

- Hilbert space : $\mathcal{H}$
- density operator (state of system) : ρ
- self-ajoint operator on $\mathcal{H}$ (observable) : X
- bounded Borel function on $\mathbb{R}$: f

$$\text{Expectation of } f(X):\ \mathrm{Tr}(\rho f(X)) = \int_{\mathbb{R}} f(x)\,d\mu(x)$$

$$\text{probability of } X \in E \in \mathrm{Bor}(\mathbb{R}):\ \mathrm{Tr}(\rho \mathbb{1}_E(X)) = \mu(E)$$

Objective

1. Quantum analogues of σ-algebra and filtrations in classical probability
2. Transformation mechanism between observables and random variable

1. S. Attal, "*Quantum noise theory*", http://math.univ-lyon1.fr/~attal/chapters.html

von Neumann algebra and normal state

Definition : $*$-algebra on $\mathcal{H}$

A $*$-algebra on $\mathcal{H}$ is a collection $\mathcal{A}$ of linear operators on $\mathcal{H}$ containing $\mathbb{1}$ s.t.

1. $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$ implies $\alpha A + \beta B \in \mathcal{A}$.
2. $A, B \in \mathcal{A}$ implies $AB \in \mathcal{A}$.
3. $A \in \mathcal{A}$ implies $A^* \in \mathcal{A}$, where the mapping $*$ is called an involution.

Moreover, $\mathcal{A}$ is called *commutative* if $AB = BA$ for any $A, B \in \mathcal{A}$.

Definition : state

A state φ on $*$-algebra $\mathcal{A}$ is a functional $\varphi : \mathcal{A} \to \mathbb{C}$ s.t.

1. linearity : $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$ implies $\varphi(\alpha A + \beta B) = \alpha\varphi(A) + \beta\varphi(B)$.
2. positivity : for all $A \geq 0$ in $\mathcal{A}$, $\varphi(A) \geq 0$.
3. normalization : $\varphi(\mathbb{1}) = 1$.

von Neumann algebra and normal state

Definition : von Neumann algebra

A von Neumann algebra on $\mathcal{H}$ is a $*$-subalgebra of $\mathscr{B}(\mathcal{H})$, containing the identity $\mathbb{1}$ and strongly closed, i.e., $A_i \in \mathcal{A}$ and $\forall \psi \in \mathcal{H}$, $\lim_{i\to\infty} A_i\psi = A\psi$ implies $A \in \mathcal{A}$.

Definition : faithful and normal state

A state φ on von Neumann algebra $\mathcal{A}$ is called

- *faithful*, if $\varphi(A^*A) = 0$ implies $A = 0$;
- *normal*, if $\varphi(\sup_\alpha A_\alpha) = \sup_\alpha \varphi(A_\alpha)$ for all bounded increasing net A_α.

Quantum probability space

Definition : quantum probability space

A quantum probability space is a pair $(\mathcal{A}, \varphi)$, where

- $\mathcal{A}$ is a von Neumann algebra (on $\mathcal{H}$) ;
- φ is a normal state on $\mathcal{A}$.

Proposition [1]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then

- $\mathcal{A} := \{M_f | f \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})\}$ is a commutative von Neumann algebra of the operators on $L^2(\Omega, \mathcal{F}, \mathbb{P})$;
- $\varphi : M_f \mapsto \int f d\mathbb{P}$ is a normal state on $\mathcal{A}$.

Remark : Classical probability space is a special case of quantum probability space.

1. H. Maassen, "*Quantum probability, Prop 1.1* ", Quantum Prob. Commu.(Vol 12), 2003.

Quantum probability space

- self-adjoint set $\mathcal{S}$: $S \in \mathcal{S} \Rightarrow S^* \in \mathcal{S}$
- commutant of $\mathcal{S}$: $\mathcal{S}' := \{X \in \mathscr{B}(\mathcal{H}) | XS = SX, \forall S \in \mathcal{S}\}$

Theorem : double commutant theorem [1]

Let $\mathcal{S} \subset \mathscr{B}(\mathcal{H})$ be any self-adjoint set. Then $\mathcal{A} = \mathcal{S}''$ is the **smallest** von Neumann subalgebra of $\mathscr{B}(\mathcal{H})$ containing $\mathcal{S}$. $\mathcal{B}$ is a von Neumann algebra iff $\mathcal{B} = \mathcal{B}''$

Corollary

The von Neumann algebra generated by $\mathcal{S} \subset \mathscr{B}(\mathcal{H})$ is $\mathrm{vN}(\mathcal{S}) := (\mathcal{S} \cup \mathcal{S}^*)''$, where $\mathcal{S}^* := \{X \in \mathscr{B}(\mathcal{H}) | X^* \in \mathcal{S}\}$.

Corollary

Given a commuting set of observables $\mathcal{X} = \{X_1, \ldots, X_n\}$, $\mathrm{vN}(\mathcal{X})$ is a commutative von Neumann algebra. ($\mathcal{X} \subset \mathcal{X}' \Rightarrow \mathcal{X}'' \subset \mathcal{X}' = \mathcal{X}'''$)

1. R. V. Kadison, J. R. Ringrose "*Funfamentals of the theory of operator algebras, Thm 5.3.1*", Vol 1, AMS, 1983.

Spectral theorem (finite-dimensional case)

Theorem : spectral theorem (finite-dimensional case) [1]

Let $(\mathcal{A}, \varphi)$ be a commutative quantum probability space on a finite-dim Hilbert space. Then there are a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $*$-isomorphism $\iota : \mathcal{A} \to l^\infty(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\varphi(X) = \mathbb{E}^{\mathbb{P}}(\iota(X))$ for all $X \in \mathcal{A}$.

Proof.

1. finite dimensional Hilbert space : $\mathbb{C}^N$
2. $\mathcal{A}$: a commutative $*$-algebra of complex $N \times N$ matrices
3. $\exists$ a unitary matrix U s.t. $U^* X U$ is diagonal $\forall X \in \mathcal{A}$
4. $\Omega := \{1, \dots, N\}$, $\mathcal{F} := 2^\Omega$
5. define $\iota(X) : \Omega \to \mathbb{C}$ by $\iota(X)(i) = (U^* X U)_{ii} \in \mathbb{C}$ for $i \in \Omega$
6. $\mathbb{P}(E) = \varphi\big(\iota^{-1}(\mathbb{1}_E)\big)$ for all $E \in \mathcal{F}$

1. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering, Thm 2.4*", SIAM. J. Control Optim, 2007.

Spectral theorem (infinite-dimensional case)

Theorem : spectral theorem (infinite-dimensional case) [1]

Let $\mathcal{H}$ be separable and $\mathcal{A}$ be a commutative von Neumann algebra on $\mathcal{H}$. Then there exist a finite measure space $(\Omega, \mathcal{F}, \mu)$ s.t. $U\mathcal{A}U^* = L^\infty(\Omega, \mathcal{F}, \mu)$ acting on $L^2(\Omega, \mathcal{F}, \mu)$ by pointwise multiplication.

Outline of proof.

1. $\mathcal{H}$ is separable, $\mathcal{A}$ is commutative $\Rightarrow \exists A = A^* \in \mathcal{A}$ s.t. $\mathcal{A} = \mathrm{vN}(A)$
2. $\exists$ finite measure space $(\Omega, \mathcal{F}, \mu)$, bounded measurable function a on Ω, and unitary map $U : \mathcal{H} \to L^2(\Omega, \mathcal{F}, \mu)$, s.t. $(UAU^* v)(\omega) = a(\omega)v(\omega)$, for all $v \in L^2(\Omega, \mathcal{F}, \mu)$ (*spectral theorem*) [2]
3. define von Neumann algebra $\mathcal{B} := \{f(A) | f$ bounded Borel on $sp(A)\}$
4. define $*$-isomorphism $\iota : \mathcal{B} \to L^\infty(\Omega, \mathcal{F}, \mu)$ by $\iota(f(X)) = Uf(X)U^* = M_{f \circ a}$ (*functional calculus*) [2]
5. $\mathcal{B} = \mathcal{A}$. [3]

1. R. van Handel, "*Filtering, stability, and robustness , Thm B.1.13*", Ph.D. Thesis, 2007.
2. M. Reed, B. Simon, "*Method of modern mathematical physics, Ch VII.2* ", AP, 1972.
3. J. R. Ringrose, R. V. Kadison, "*Fundamentals of the theory of operator algebras, Thm 5.2.9*".

Spectral theorem (infinite-dimensional case)

Corollary [1]

Let $(\mathcal{A}, \varphi)$ be a commutative quantum probability space on separable $\mathcal{H}$. Then there exist a finite measure space $(\Omega, \mathcal{F}, \mu)$, a $*$-isomorphism $\iota : \mathcal{A} \to L^\infty(\Omega, \mathcal{F}, \mu)$, and a probability measure $\mathbb{P} \ll \mu$ s.t. $\varphi(A) = \mathbb{E}^{\mathbb{P}}\big(\iota(A)\big)$ for all $A \in \mathcal{A}$.

- $P \in \mathcal{A}$ s.t. $\varphi(P) = 0$, then $\iota(P) = 0$
- μ is to define the null set in $\mathcal{F}$, then define $\mathbb{P} \ll \mu$ by φ
- commutative probability space is equivalent to classical probability space

1. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering, Thm 3.3* ", SIAM. J. Control Optim, 2007.

Spectral representation of unbounded observable

- Hilbert space $\mathcal{H}$
- $\mathcal{P}(\mathcal{H})$ the set of orthogonal projections on $\mathcal{H}$
- **spectral measure** on $(\mathbb{R}, \mathrm{Bor}(\mathbb{R}))$ is $\xi : \mathrm{Bor}(\mathbb{R})) \to \mathcal{P}(\mathcal{H})$ s.t.
  - $\xi(\emptyset) = 0$ and $\xi(\mathbb{R}) = \mathbb{1}$
  - $\xi\left(\bigcup_i E_i\right) = \text{s-}\lim_{k\to\infty} \sum_{i=1}^{k} \xi(E_i)$ for countable sequence of disjoint set E_i
  - $\xi(E_1)\xi(E_2) = \xi(E_1 \cap E_2) = \xi(E_2)\xi(E_1)$

Theorem : von Neumann's spectral theorem [1]

For any self-adjoint operator X on $\mathcal{H}$, $\exists! \, \xi : \mathrm{Bor}(\mathbb{R}) \to \mathcal{P}(\mathcal{H})$ s.t.

$$X = \int_{\mathbb{R}} x \xi(dx).$$

Then, given any Borel function f on $\mathbb{R}$, $\int_{\mathbb{R}} f(x) d\xi(x)$ is denoted by $f(X)$.

- $\langle \psi, f(X)\psi \rangle = \int_{\mathbb{R}} f(x) \langle \psi, \xi(dx)\psi \rangle$, $\forall \psi \in \left\{ \psi \in \mathcal{H} \,\middle|\, \int_{\mathbb{R}} |f(x)|^2 \langle \psi, \xi(dx)\psi \rangle < \infty \right\}$.

1. P.Meyer, "*Quantum probability for probabilists, pp 8*", LNM (V. 1538), Springer, 1993.

Unbounded observable

- (not necessarily bounded) observable X on $\mathcal{H}$ has real spectrum
- von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$

Definition

1. X is **affiliated** to $\mathcal{A}$, $(X \eta \mathcal{A})$, if spectral measure $\xi_X(E) \in \mathcal{A}$, $\forall E \in \mathrm{Bor}(\mathbb{R})$.
2. von Neumann algebra generated by X : $\mathrm{vN}(X) := \mathrm{vN}(\{\xi_X(E) | E \in \mathrm{Bor}(\mathbb{R})\})$

- the above probabilistic definition is equivalent to the algebraic one [1]
- observable is affiliated to $\mathcal{A} \leftrightarrow$ r.v is measurable w.r.t σ-algebra
- $X \in \mathcal{B}(\mathcal{H})$, X is affiliated to $\mathcal{A}$ iff $X \in \mathcal{A}$ [2],
- $X \in \mathcal{B}(\mathcal{H})$, $\mathrm{vN}(X) = \mathrm{vN}(\{\xi_X(E) | E \in \sigma\text{-algebra on } sp(X)\})$ and $X \eta \, \mathrm{vN}(X)$

1. P.Meyer, "*Quantum probability for probabilists, pp 245*", LNM (Vol 1538), Springer, 1993.
2. J. R. Ringrose, R. V. Kadison, "*Fundamentals of the theory of operator algebras, Thm 5.2.3*", Vol 1, AMS, 1983.

Unbounded affiliated observable

- $X \,\eta\, \mathcal{A}$, $\mathcal{A}$ is commutative $\Rightarrow \exists A = A^* \in \mathcal{A}$ s.t. $\mathcal{A} = \mathrm{vN}(A)$
- $(X + i\mathbb{1})^{-1}$ is bounded and belongs to $\mathrm{vN}(A)$
- **spectral theorem** on $\mathcal{A} \Rightarrow U : \mathcal{H} \to L^2(\Omega, \mathcal{F}, \mu)$ and $*$-isomorphism $\iota : \mathcal{A} \to L^\infty(\Omega, \mathcal{F}, \mu) \Rightarrow \iota\big((X + i\mathbb{1})^{-1}\big)$
- *spectral theorem for unbounded observable*[1] implies

$$\iota(X)(\omega) = \frac{1}{\iota\big((X + i\mathbb{1})^{-1}\big)(\omega)} - 1, \quad \omega \in \Omega,$$

$\iota(X)$ is a μ-a.s. finite $\mathcal{F}$-measurable r.v. on Ω.

1. M. Reed, B. Simon, "*Method of modern mathematical physics, Thm VIII.4*", AP, 1972.

Additive and multiplication of affiliated observables [1]

- commutative von Neumann algebra : $\mathcal{A}$
- **spectral theorem** on $\mathcal{A} \Rightarrow *$-isomorphism $\iota : \mathcal{A} \to L^\infty(\Omega, \mathcal{F}, \mu)$
- set of all self-adjoint operators affiliated to $\mathcal{A}$: $\mathscr{S}(\mathcal{A})$

Lemma

For any $X, Y \,\eta\, \mathcal{A}$, $X \hat{+} Y := \overline{X+Y}$ and $X \hat{\cdot} Y := \overline{XY}$ are self-adjoint and affiliated to $\mathcal{A}$

Lemma

- $\mathscr{S}(\mathcal{A})$ forms a commutative $*$-algebra (with unit $\mathbb{1}$) under $\hat{+}$ and $\hat{\cdot}$.
- $\iota : \mathcal{A} \to L^\infty(\Omega, \mathcal{F}, \mu)$ extends to an isomorphism between $\mathscr{S}(\mathcal{A})$ and the set of μ-a.s. finite $\mathcal{F}$-measurable r.v. on Ω.

1. J. R. Ringrose, R. V. Kadison, "*Fundamentals of the theory of operator algebras, pp 351-356*", Vol 1, AMS, 1983.

Additive and multiplication of affiliated observables [1]

- commutative von Neumann algebra : $\mathscr{A}$
- **spectral theorem** on $\mathscr{A} \Rightarrow *$-isomorphism $\iota : \mathscr{A} \to L^\infty(\Omega, \mathcal{F}, \mu)$
- set of all normal operators affiliated to $\mathscr{A}$: $\mathscr{N}(\mathscr{A})$

Lemma

- A closed and densely defined operator X is **normal** if $X \hat{+} X^*$ and $i(X^* \hat{-} X)$ are self-adjoint and commute with each other.
- A normal operator X is affiliated to $\mathscr{A}$ if $X \hat{+} X^*$ and $i(X^* \hat{-} X)$ are affiliated to $\mathscr{A}$.

Lemma

- $\mathscr{N}(\mathscr{A})$ forms a commutative $*$-algebra (with unit $\mathbb{1}$) under $\hat{+}$ and $\hat{\cdot}$.
- $\iota : \mathscr{A} \to L^\infty(\Omega, \mathcal{F}, \mu)$ extends to an isomorphism between $\mathscr{N}(\mathscr{A})$ and the set of μ-a.s. finite $\mathcal{F}$-measurable random variable on Ω.

1. J. R. Ringrose, R. V. Kadison, "*Fundamentals of the theory of operator algebras, pp 351-356*", Vol 1, AMS, 1983.

Example : position and momentum operators

- Schrödinger representation on Schwartz space $\mathscr{S}(\mathbb{R})$
$$(Q\psi)(x) = x\psi(x), \quad (P\psi)(x) = -i\hbar\frac{d}{dx}\psi(x), \quad \psi \in \mathcal{H},$$
- P and Q are defined as closures of $i^{-1}d/dx$ and multiplication by x on $\mathscr{S}(\mathbb{R})$, Q and P are self-adjoint
- $\psi(x) = (2\pi)^{-1/4}\sigma^{-1/2}e^{-\frac{(x-\mu)^2}{4\sigma^2}}$ defines a normal state on $\mathscr{B}(\mathcal{H})$
- $\forall E \in \mathrm{Bor}(\mathbb{R})$, $(\xi_Q(E)\psi)(x) = \mathbb{1}_E(x)\psi(x)$.
- $\mathrm{vN}(Q) = L^\infty(\mathbb{R})$
- $\forall E \in \mathrm{Bor}(\mathbb{R})$, $\mathbb{P}_Q\big(\iota(Q) \in E\big) = \varphi\big(\xi_Q(E)\big) = \int_E \psi^2(x)dx$ is a **Gaussian** measure with mean μ and variance σ^2.

Example : position and momentum operators

$$\boxed{\begin{aligned}\mathbb{E}\big(\iota(e^{itQ})\big) &= \langle\psi, e^{itQ}\psi\rangle = \int_{\mathbb{R}} e^{itx}\psi^2(x)dx = e^{it\mu - \frac{t^2\sigma^2}{2}},\\ \mathbb{E}\big(\iota(e^{itP})\big) &= \langle\psi, e^{itP}\psi\rangle = \int_{\mathbb{R}} \psi(x)\psi(x+\hbar t)dx = e^{-\frac{\hbar^2 t^2}{8\sigma^2}}.\end{aligned}}$$

- $\iota(Q) \sim \mathcal{N}(\mu, \sigma^2)$
- $\iota(P) \sim \mathcal{N}(0, \hbar^2/4\sigma^2)$

Quantum stochastic process

Quantum stochastic process : Wiener and Poisson processes

References :

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus*", Birkhauser, 1992.
2. L. Bouten, R. van Handel, M. James, "*An introduction to quantum filtering*", SIAM. J. Control Optim, 2007.
3. L. Bouten, "*Applications of quantum stochastic processes in quantum optics*", Quantum Potential Theory, pp 277-307, Springer, 2008.
4. R. van Handel, "*Filtering, stability, and robustness*", Ph.D. Thesis, 2007.

Fock space

- For $u_1, \ldots, u_N \in \mathcal{H}$, $u_1 \circ \cdots \circ u_N := \frac{1}{N!} \sum_{\sigma \in \mathscr{P}_N} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(N)}$, where $\mathscr{P}_N$ is permutation group on N elements.
- $\mathcal{H}^{\circ N}$: the closed subspace of $\mathcal{H}^{\otimes N}$ generated by all vectors $u_1 \circ \cdots \circ u_N$
- scalar products defined on $\mathcal{H}^{\otimes N}$ and $\mathcal{H}^{\circ N}$

$$\langle u_1 \otimes \cdots \otimes u_N, v_1 \otimes \cdots \otimes v_N \rangle_{\otimes} = \langle u_1, v_1 \rangle \ldots \langle u_N, v_N \rangle;$$
$$\langle u_1 \circ \cdots \circ u_N, v_1 \circ \cdots \circ v_N \rangle_{\circ} = \mathrm{Per}\big(\langle u_i, v_j \rangle\big)_{0 \le i,j \le N},$$

Definition : symmetric Fock space

A symmetric (or bosonic) Fock space over $\mathcal{H}$ is $\Gamma_s(\mathcal{H}) := \mathbb{C} \oplus \bigoplus_{n=1}^{+\infty} \mathcal{H}^{\circ n}$, $\mathcal{H}$ is called single-particle Hilbert space.

Remark : $\Gamma_s(\mathcal{H})$ is a separable Hilbert space if $\mathcal{H}$ is separable.

Exponential vector

- **exponential vector** : $e(u) = \bigoplus_{n=0}^{+\infty} \frac{u^{\otimes n}}{\sqrt{n!}} \in \Gamma_s(\mathcal{H})$ with $u \in \mathcal{H}$
- **vacuum vector** : $e(0) = 1 \oplus 0 \oplus 0 \oplus \dots$
- $\langle e(u), e(v) \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \langle u^{\otimes n}, v^{\otimes n} \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \big(\langle u, v \rangle \big)^n = e^{\langle u,v \rangle}$
- **exponential domain** : $\mathcal{E}(\mathcal{H}) := \mathrm{span}\{e(u) | \, u \in \mathcal{H}\}$
- $\mathcal{E}(\mathcal{H})$ is **dense** in $\Gamma_s(\mathcal{H})$, the generators $e(u)$ of $\mathcal{E}(\mathcal{H})$ are linearly independent [1]

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, pp 126-127*", Birkhauser, 1992.

Stone's theorem

Definition : strongly continuous one-parameter unitary group

An operator-valued function U_t on $\mathcal{H}$ is called a **strongly continuous one-parameter unitary group** if it satisfies

1. group property : $\forall s, t \in \mathbb{R}$, $U_{t+s} = U_t U_s$;
2. strong continuity : $\forall t_0 \in \mathbb{R}$ and $\psi \in \mathcal{H}$, $\lim_{t \to t_0} U_t \psi = U_{t_0} \psi$.

Theorem : Stone's theorem

Let $\{U_t\}_{t \in \mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathcal{H}$. Then there is a self-adjoint operator A on $\mathcal{H}$ s.t. $U_t = e^{itA}$.

Weyl operators

Theorem : Weyl operators [1]

- For any $u, v \in \mathcal{H}$ and unitary operator U on $\mathcal{H}$, there exists an unique *unitary operator* (**Weyl operator**) $W(u, U)$ on $\Gamma_s\big(L^2(\mathbb{R}_+)\big)$ satisfying

$$W(u, U)e(v) = e^{-\langle u, Uv\rangle - \|u\|^2/2} e(Uv + u).$$

- For any $u_i, v \in \mathcal{H}$ and unitary operator U_i on $\mathcal{H}$ with $i = 1, 2$,

$$W(u_1, U_1)W(u_2, U_2) = e^{-i\mathrm{Im}\langle u_1, U_1 u_2\rangle} W\big(u_1 + U_1 u_2, U_1 U_2\big).$$

- $(u_1, U_1) = (u, \mathbb{1})$ and $(u_2, U_2) = (0, U)$, $W(u, U) = W(u, \mathbb{1})W(0, U)$
- $W_u := W(u, \mathbb{1})$, $\Gamma(U) := W(0, U)$

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, pp 135*", 1992.

Field operator and differential second quantization

Translation group by the vector u :

1. Weyl relation : $W_u^* = W_{-u}$, $W_u W_v = e^{-i\mathrm{Im}\langle u,v\rangle} W_{u+v}$
2. $W_{su} W_{tu} = W_{(s+t)u}$ and its strong continuity [1]
3. W_{tu} for $t \in \mathbb{R}$ forms a one-parameter strong continuous unitary group
4. $\forall u \in \mathcal{H}$, $\exists B(u)$ (**field operator**) s.t. $W_{tu} = e^{itB(u)}$, $\forall t \in \mathbb{R}$ (Stone's theorem)
5. $B(u)$ and $B(v)$ commute if $\langle u, v\rangle \in \mathbb{R}$ [2]

Rotation group by the unitary operator U :

1. $\Gamma(U)e(v) = e(Uv) \Rightarrow \Gamma(U)\Gamma(V) = \Gamma(UV)$
2. given $U_t = e^{itA}$, $\Gamma(U_t)$ defines a one-parameter strongly continuous unitary group on $\Gamma_s(\mathcal{H})$
3. $\forall u \in \mathcal{H}$, $\exists \Lambda(A)$ (**differential second quantization of** A) s.t. $\Gamma(U_t) = e^{it\Lambda(A)}$, $\forall t \in \mathbb{R}$ (Stone's theorem)
4. $\Lambda(A_1)$ and $\Lambda(A_2)$ commute if $[A_1, A_2] = 0$ [2]

1. K. R. Parthasarthy, "*An introduction to quantum stochastic calculus, Prop 20.1*", 1992.
2. M. Reed, B. Simon, "*Method of modern mathematical physics, Thm VIII.13* ", AP, 1972.

Quantum fundamental stochastic processes

Definition : fundamental stochastic processes

- quadratures : $Q_t = B(i\mathbb{1}_{[0,t]})$, $P_t = B(-\mathbb{1}_{[0,t]})$
- **gauge process** : $\Lambda_t = \Lambda(M_{\mathbb{1}_{[0,t]}})$, $M_{\mathbb{1}_{[0,t]}} f = \mathbb{1}_{[0,t]} f$ for $f \in L^2(\mathbb{R}_+)$

Remark : Q_t, P_t, Λ_t are commutative, they do not commute with each other

Definition : filtration

$\mathcal{Q}_t := \mathrm{vN}\{Q_s | 0 \le s \le t\}$, $\mathcal{P}_t := \mathrm{vN}\{P_s | 0 \le s \le t\}$, $\Lambda_t := \mathrm{vN}\{\Lambda_s | 0 \le s \le t\}$

1. **vacuum state** Φ on $\mathscr{B}(\Gamma_s(L^2(\mathbb{R}_+)))$: $\Phi = \langle e(0), \cdot\, e(0)\rangle$
2. **coherent state** Φ_f : $\Phi_f = \langle W_f e(0), \cdot\, W_f e(0)\rangle = e^{-\|f\|^2}\langle e(f), \cdot\, e(f)\rangle$
3. commutative quantum probability space $(\mathcal{Q}_t, \Phi)$, $(\mathcal{P}_t, \Phi)$, (Λ_t, Φ_f)
4. spectral theorem $\Rightarrow \iota(Q_t)$, $\iota(P_t)$, $\iota(\Lambda_t)$ on different probability spaces

Wiener processes in vacuum state

Lemma

$q_t = \iota(Q_t)$ and $p_t = \iota(P_t)$ are Wiener processes in vacuum state.

Proof.

1. For $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4 < \infty$, $x, y \in \mathbb{R}$, the joint characteristic function of $q_{t_4} - q_{t_3}$ and $q_{t_2} - q_{t_1}$:
$$\mathbb{E}^{\mathbb{P}_0}\left(e^{ix(q_{t_4}-q_{t_3})+iy(q_{t_2}-q_{t_1})}\right) = \langle e(0), W_{ix\mathbb{1}_{[t_3,t_4]}+iy\mathbb{1}_{[t_1,t_2]}} e(0)\rangle$$
$$= e^{-\left\|x\mathbb{1}_{[t_3,t_4]}+y\mathbb{1}_{[t_1,t_2]}\right\|^2/2} = e^{-x^2(t_4-t_3)/2} e^{-y^2(t_2-t_1)/2}.$$
2. q_t has independent increments, $q_t - q_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s \leq t < \infty$.
3. Kolmogorov continuity theorem, there exist a continuous modification of q_t.
4. proof for p_t is identical.

Poisson process in coherent state

Lemma

$\lambda_t = \iota(\Lambda_t)$ is

- a.s. zero in vacuum state
- Poisson process with time-dependent intensity $|f(t)|^2$ in coherent state Φ_f.

Proof.

1. $\Gamma(U)e(0) = e(0)$, $\iota(\Lambda_t)$ is a.s. zero in vacuum state
2. for $0 \le t_1 \le t_2 \le t_3 \le t_4 < \infty$, $x, y \in \mathbb{R}$, the joint characteristic function of $\lambda_{t_4} - \lambda_{t_3}$ and $\lambda_{t_2} - \lambda_{t_1}$:
$$\mathbb{E}^{\mathbb{P}}\left(e^{ix(\lambda_{t_4}-\lambda_{t_3})+iy(\lambda_{t_2}-\lambda_{t_1})}\right) = e^{-\|f\|^2}\langle e(f), \Gamma(e^{ixM_{\mathbb{1}_{[t_3,t_4]}}} e^{iyM_{\mathbb{1}_{[t_1,t_2]}}})e(f)\rangle$$
$$= e^{-\|f\|^2}\exp\left(\langle f, e^{ixM_{\mathbb{1}_{[t_3,t_4]}}} e^{iyM_{\mathbb{1}_{[t_1,t_2]}}} f\rangle\right)$$
$$= \exp\left(\int_{t_3}^{t_4} |f(t)|^2 dt(e^{ix}-1)\right)\exp\left(\int_{t_1}^{t_2} |f(t)|^2 dt(e^{iy}-1)\right).$$
3. $\lambda_t := \iota_f(\Lambda_t)$ has independent increments, $\lambda_t - \lambda_s \sim \mathrm{Pois}(\int_s^t |f(x)|^2 dx)$ for $0 \le s \le t < \infty$.

Wiener and Poisson processes in quantum optics [1]

In **quantum optics** :

- Q_t and P_t can be observed by using a **homodyne detector** to measure the vacuum
- Λ_t can be measured by a **photon counter**

1. A. Barchielli, "*Continual measurements in quantum mechanics and quantum stochastic calculus*", Open Quantum Systems III, 2006.