# Introduction to quantum feedback control

CY-McGill Mathematical Physics Weekly Seminar

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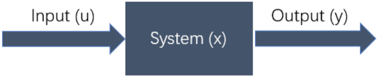
# Outline

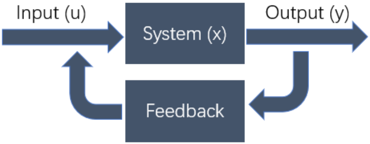
# Classical feedback control

**Control objective :**

- Stabilize the system towards a target state (stabilization) ;
- Minimize a cost function (optimal control).




Open-loop control

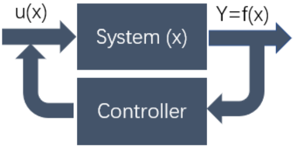



Closed-loop control

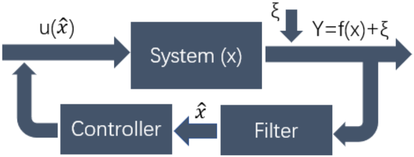
- Control input is predetermined, no feedback is involved.

- Control input depends on the information through the system measurements. (Robust)

# Classical feedback control




Feedback control with complete observations




Feedback control with **partial** observations

- $\hat{x}_t = \mathbb{E}(x_t | \sigma(y_{s\leq t}))$ (MMSE), linear function $\pi_t(I) = \mathbb{E}(I(x_t)|\sigma(y_{s\leq t})) = I(\hat{x}_t)$
- $\pi_t(h) = \mathbb{E}(h(x_t)|\sigma(y_{s\leq t}))$ : best estimation of $h(x_t)$ given $y_{s\leq t}$ in $L^2$ (filtering)

# Quantum mechanics in finite-dimensional setting

- **Density operator :** $\rho = \rho^* \in \mathbb{C}^{N\times N}$, $\mathrm{Tr}(\rho) = 1$ and $\rho \geq 0$
- **Observable :** $X = X^* \in \mathbb{C}^{N\times N}$
- **Evolution :** closed quantum system

$$\begin{aligned} \text{Schrödinger P. :} \quad & \dot{\rho}(t) = -i[H, \rho(t)], \quad \rho(t) = U(t)\rho(0)U^*(t); \\ \text{Heisenberg P. :} \quad & \dot{X}(t) = i[H, X(t)], \quad X(t) = U^*(t)X(0)U(t), \end{aligned}$$

where $H = H^*$, $U(t)U^*(t) = \mathbb{1}$ and $\mathrm{Tr}(X(t)\rho(0)) = \mathrm{Tr}(X(0)\rho(t))$.

# Quantum feedback control

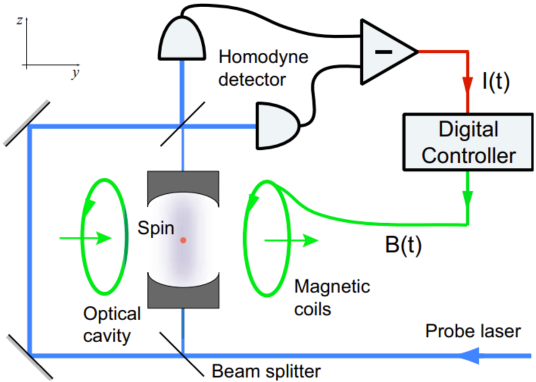



FIGURE – Experiment setup for feedback control of spin system, which interacts with an optical field measured continuously by homodyne detection. A magnetic field is used for the feedback [1].

1. R. van Handel, J.K. Stockton, H.Mabuchi, IEEE TAC, 2005.

# Problems on quantum feedback control

1. How to model the system-field interaction ?
2. How to model the continuous measurement of the field ?
3. How to estimate the state of the system based on the measurements ?
4. How to design a feedback controller to achieve a control goal ?

# Lecture outline

1. Input-output model, classical filtering theory ;
2. Quantum probability theory, quantum filtering theory ;
3. Stochastic control theory, literature reviews [2] [3] ;
4. Exponential feedback stabilization of qubit / 2-qubit systems ;
5. Exponential feedback stabilization of spin-J / N-qubit systems ;
6. Robustness of stabilizing qubit systems (unknown initial states) ;
7. Robustness of stabilizing spin-J systems (unknown initial states) ;
8. Discussion on insufficient computing power.

1. R. van Handel, J.K. Stockton, H.Mabuchi, "*Feedback control of quantum state reduction*", IEEE TAC, 2005.
2. M. Mirrahimi, R. van Handel, "*Stabilizing feedback controls for quantum systems*", SIAM J Control Optim, 2007.

# Brief history on quantum feedback control

1. **Belavkin** [1] (1970s) : quantum analogous of stochastic control theory, Belavkin quantum filtering equation (estimation).
2. **Hudson, Parthasarathy** [2] (1984) : quantum stochastic calculus and quantum Itô formula.
3. **Gardiner, Collett** [3] (1985) : quantum analogous of input-output model, quantum Langevin equation.
4. **Carmichael** [4] et al. (1990s) : quantum trajectory theory (simulation).
5. **Bouten, van Handel, James** [5] (2007) : morden formulation of Belavin's work.
6. **Serge Haroche, David Wineland** : Nobel Prize in Physics in 2012.

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1. https://www.maths.nottingham.ac.uk/plp/vpb/vpb_research.html
2. R. L. Hudson, K. R. Parthasarathy, "*Quantum Ito's formula and stochastic evolutions*", Commun. Math Phys, 1984.
3. C. W. Gardiner, M. J. Collett, "*Input and output in damped quantum systems : Quantum stochastic differential equation and master equation*", PRA, 1985.
4. H. Carmichael, "*An Open Systems Approach to Quantum Optics*", Springer, 1993.
5. L. Bouten, R. van Handel, M. James,"*An introduction to quantum filtering*", SIAM. J. Control Optim, 2007.

# Open quantum system and Input-output model

**Open quantum system and Input-output model**

# Open quantum system and master equation

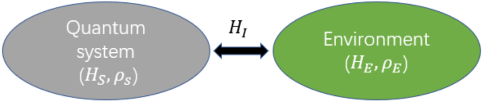



FIGURE – Open quantum system : a quantum system interacting with an external environment (a gas of particles, a heat bath, a beam of photons, etc.)

- **Hamiltonian approach** : $H_{tot} = H_S \otimes \mathbb{1}_E + \mathbb{1}_S \otimes H_E + H_I$.
- **Markovian approach** : focus on dynamics of quantum system
- **Master equation method** : describe dynamics of quantum system by tracing out degrees of freedom of environment

# Open quantum system and master equation [1]

1. system-environment state : $\rho$ on $\mathcal{H}_S \otimes \mathcal{H}_E$
2. partial trace : $\mathrm{Tr}\big(\mathrm{Tr}_{\mathcal{H}_E}(\rho) X_S\big) = \mathrm{Tr}\big(\rho(X_S \otimes \mathbb{1}_E)\big)$, $X_S$ on $\mathcal{H}_S$
3. quantum state (marginal) : $\rho_S = \mathrm{Tr}_{\mathcal{H}_E}(\rho)$
4. initial state (uncorrelated) : $\rho(t_0) = \rho_S(t_0) \otimes \rho_E(t_0)$
5. time evolution : $\rho(t) = U(t, t_0)\big(\rho_S(t_0) \otimes \rho_E(t_0)\big) U^*(t, t_0)$
6. time evolution of quantum state : $\rho_S(t) = \mathrm{Tr}_{\mathcal{H}_E}\big(\rho(t)\big)$
7. Born-Markov approx : weak coupling + environment short memory
8. **Master equation** : $\frac{d}{dt}\rho_S(t) = \mathcal{L}(\rho_S(t))$, with **Lindblad generator**

$$\mathcal{L}(\rho_S) = i[H_S, \rho_S] + \sum_i \left(L_i \rho_S L_i^* - \frac{1}{2} L_i^* L_i \rho_S - \frac{1}{2}\rho_S L_i^* L_i\right).$$

1. H. M. Wiseman, G. J. Milburn, "*Quantum measurement and control, Ch3*", Cambridge, 2009.

# Input-output model for Markov quantum systems

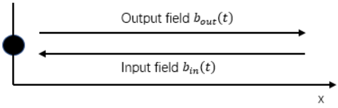



FIGURE – A quantum system weakly coupled to a single electromagnetic field

**Motivation :**

- allow to calculate the output field
- connect field and continuous measurements

**Electromagnetic field :** a collection of quantum harmonic oscillators

- annihilation operator $b_\omega$, creation operator $b_\omega^*$
- CCR : $[b_\omega, b_{\omega'}] = 0$ and $[b_\omega, b_{\omega'}^*] = \delta(\omega - \omega')$
- h.o. Hamiltonian : $H_\omega = \omega b_\omega^* b_\omega$
- field Hamiltonian : $H_E = \int_{-\infty}^{\infty} \omega b_\omega^* b_\omega d\omega$

# Input-output model for Markov quantum systems

- Total Hamiltonian : $H_{tot} = H_S + H_E + H_I$,

$$H_E = \int_{-\infty}^{\infty} \omega b_\omega^* b_\omega d\omega, \quad H_I = i\int_{-\infty}^{\infty} \kappa(\omega)[b_\omega^* C - b_\omega C^*] d\omega \quad \text{(RWA)},$$

with $C$ : system operator, $\kappa(\omega) \in \mathbb{R}$ : coupling constant.

- Time evolution of $b_\omega$ in H.P.

$$\frac{d}{dt} b_\omega(t) = i[H_E + H_I, b_\omega(t)] = -i\omega b_\omega(t) + \kappa(\omega) C(t),$$

$$b_\omega(t) = e^{-i\omega t} b_\omega + \kappa(\omega) \int_0^t e^{-i\omega(t-s)} C(s) ds, \quad \text{(not Markovian)}$$

with $b_\omega(0) = b_\omega$, $C(0) = C$, $C(t)$ : time evolution of $C$ in H.P.

- Time evolution of system observable $X$ in H.P.

$$\frac{d}{dt} X(t) = i[H_S + H_I, X(t)]$$
$$= i[H_S, X(t)] + \int_{-\infty}^{\infty} \kappa(\omega) \big( b_\omega^*(t)[X(t), C(t)] - [X(t), C^*(t)] b_\omega(t) \big) d\omega,$$

with $X(0) = X$ and $C(0) = C$.

# Input-output model for Markov quantum systems

- **First Markov approximation** : $\kappa(\omega) = \sqrt{\gamma/2\pi}$
- **Input field** : $b_{in}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} b_\omega d\omega$ satisfies
$$[b_{in}(s), b_{in}(t)] = 0, \quad [b_{in}(s), b_{in}^*(t)] = \delta(s-t).$$
- $\int_{-\infty}^{\infty} e^{-i\omega t} d\omega = 2\pi\delta(t)$ and $\int_0^t C(s)\delta(t-s)ds = C(t)/2$
- **Quantum Langevin equation** :
$$\frac{d}{dt}X(t) = +i[H_S, X(t)] + \sqrt{\gamma}\big(b_{in}^*(t)[X(t), C(t)] - [X(t), C^*(t)]b_{in}(t)\big) + \gamma\big(C^*(t)X(t)C(t) - \tfrac{1}{2}C^*(t)C(t)X(t) - \tfrac{1}{2}X(t)C^*(t)C(t)\big)$$
- $b_\omega(t) = e^{-i\omega t} b_\omega(T) - \kappa(\omega)\int_t^T e^{-i\omega(t-s)} C(s)ds$, for $t < T$
- **Output field** : $b_{out}(t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(t-T)} b_\omega(T) d\omega = \sqrt{\gamma}C(t) + b_{in}(t)$

# Quantum Langevin equation implies Master equation

**In vacuum state** $|0\rangle$ :

- $b_{in}(t)|0\rangle = \langle 0|b^*_{in}(t) = 0$, $\langle 0|0\rangle = 1$,

- White noise : $x(t) := b_{in}(t) + b^*_{in}(t)$ and $y(t) := ib_{in}(t) - ib^*_{in}(t)$

$$\langle x(t)\rangle := \langle 0|x(t)|0\rangle = 0, \langle x(t)x(s)\rangle = \delta(t-s);$$
$$\langle y(t)\rangle := \langle 0|y(t)|0\rangle = 0, \langle y(t)y(s)\rangle = \delta(t-s)$$

- $\rho_E = |0\rangle\langle 0|$ and $\mathrm{Tr}(X(t)\rho_S \otimes \rho_E) = \mathrm{Tr}(\bar{X}(t)\rho_S)$ implies

$$\frac{d}{dt}\bar{X}(t) = +i[H_S, \bar{X}(t)] + \gamma\big(C^*(t)\bar{X}(t)C(t) - \tfrac{1}{2}C^*(t)C(t)\bar{X}(t) - \tfrac{1}{2}\bar{X}(t)C^*(t)C(t)\big)$$

- $\mathrm{Tr}(\bar{X}(t)\rho_S) = \mathrm{Tr}(\bar{X}\rho_S(t))$ implies **Master equation**

$$\frac{d}{dt}\rho_S(t) = i[H_S, \rho_S(t)] + \gamma\big(C\rho_S(t)C^* - \tfrac{1}{2}C^*C\rho_S(t) - \tfrac{1}{2}\rho_S(t)C^*C\big).$$

# Input-output model undergoing homodyne detection

**System-observation model** (partial observations) :

$$\frac{d}{dt}X(t) = +i[H_S, X(t)] + \gamma\big(C^*(t)X(t)C(t) - \tfrac{1}{2}C^*(t)C(t)X(t) - \tfrac{1}{2}X(t)C^*(t)C(t)\big) + \sqrt{\gamma}\big(b_{in}^*(t)[X(t), C(t)] - [X(t), C^*(t)]b_{in}(t)\big)$$

$$\frac{d}{dt}Y_t = \frac{1}{\sqrt{\gamma}}\big(b_{out}(t) + b_{out}^*(t)\big) = \big(C(t) + C^*(t)\big) + \frac{1}{\sqrt{\gamma}}\big(b_{in}(t) + b_{in}^*(t)\big)$$

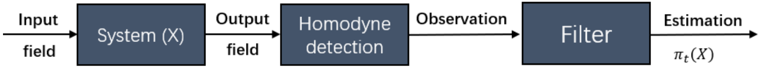



FIGURE – Diagram of quantum filtering setup

- **Quantum probability theory** (conditional expectation, stochastic process)
- **Quantum filtering theory** (explicit expression of $\pi_t(X)$)
- **Stochastic master equation** (matrix-valued stochastic differential equation)

# Classical non-linear filtering theory

**Classical non-linear filtering theory**

# Classical probability theory

- **Probability space** $(\Omega, \mathcal{F}, \mathbb{P})$
  - $\Omega$ : sample space, the set of all possible outcomes ;
  - $\mathcal{F}$ : $\sigma$-algebra of subsets of $\Omega$, a set of events ;
  - $\mathbb{P} : \mathcal{F} \to [0,1]$ : probability measure on $\mathcal{F}$.
- Real-valued **random variable** $X : \Omega \to \mathbb{R}$, $X^{-1}(E) \in \mathcal{F}$ for all $E \in \mathcal{R}$.
- **Expectation** of an integrable r.v. $X$ : $\mathbb{E}(X) = \int_{\omega\in\Omega} X(\omega) d\mathbb{P}(\omega)$.

**Theorem (Conditional expectation)**

*Suppose $X$ is an integrable r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$. Then there exists a r.v. $\mathbb{E}(X|\mathcal{G})$ called the* ***conditional expectation*** *of $X$ given $\mathcal{G}$ s.t.*

1. $\mathbb{E}(X|\mathcal{G})$ *is $\mathcal{G}$-measurable ;*
2. *for all $G \in \mathcal{G}$, $\mathbb{E}(X\mathbb{1}_G) = \mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\mathbb{1}_G\big)$, where*

$$\mathbb{E}(X\mathbb{1}_G) = \int_G X(\omega) d\mathbb{P}(\omega), \quad \mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\mathbb{1}_G\big) = \int_G \mathbb{E}(X|\mathcal{G})(\omega) d\mathbb{P}(\omega).$$

# Properties of conditional expectation

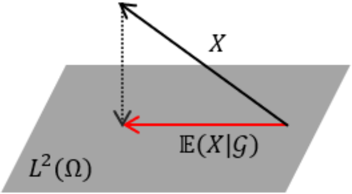
1. if $X$ is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$;
2. linearity: for all $\alpha, \beta \in \mathbb{R}$, $\mathbb{E}(\alpha X + \beta Y|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(Y|\mathcal{G})$;
3. stability: if $X$ is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$;
4. module property: if $X$ is $\mathcal{G}$-measurable, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$;
5. tower property: if $\mathcal{E} \subset \mathcal{G} \subset \mathcal{F}$, then $\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})|\mathcal{E}\big) = \mathbb{E}(X|\mathcal{E})$;
6. law of total expectation: $\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\big) = \mathbb{E}(X)$.

# Optimal estimation

**Lemma (Optimal estimation)**

Let $X$ be an integrable r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$. Then $\mathbb{E}(X|\mathcal{G})$ is the unique $\mathcal{G}$-measurable r.v. satisfying

$$\mathbb{E}\left(\left(X - \mathbb{E}(X|\mathcal{G})\right)^2\right) = \min_{Y \in L^2(\Omega,\mathcal{G},\mathbb{P})} \mathbb{E}\left((X-Y)^2\right).$$



# Bayes formula

## Theorem (Bayes formula)

*Suppose that $X$ is an integrable r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{G} \subset \mathcal{F}$. Let $\mathbb{Q} \gg \mathbb{P}$*[1] *be another probability measure such that $M = d\mathbb{P}/d\mathbb{Q}$*[2]*. Then*

$$\mathbb{E}(X|\mathcal{G}) = \frac{\mathbb{E}^{\mathbb{Q}}(XM|\mathcal{G})}{\mathbb{E}^{\mathbb{Q}}(M|\mathcal{G})}, \quad \mathbb{P} - a.s.$$

1. $\mathbb{P}$ is absolutely continuous w.r.t the measure $\mathbb{Q}$.
2. Radon-Nikodym derivative

# Brownian motion

## Brownian motion

Real-valued one dimensional Brownian motion $W_t$ can be characterized by

1. $W_0 = 0$ ;
2. $W_t$ is almost surely continuous ;
3. $W_t$ has **independent** increments ;
4. $(W_t - W_s) \sim \mathcal{N}(0, t - s)$, for $0 \leq s \leq t$.

$\mathbb{R}^n$-valued process $W_t = (W_t^1, \dots, W_t^n)$ is $n$-dimensional Brownian motion if $W_t^1, \dots, W_t^n$ are **independent** Brownian motions.

# Itô formula

## Theorem (Itô formula)

*Let $X_t$ be an Itô process $dX_t = f(t,X_t)dt + g(t,X_t)dW_t$. Let $h(t,x)$ be twice continuously differentiable in $x$ and once in $t$, then $Y_t = h(t,X_t)$ is also an Itô process and*

$$dY_t = \mathscr{L}h(t,X_t)dt + \frac{\partial h}{\partial x}g(t,X_t)dW_t,$$

$$\mathscr{L}h(t,X_t) := \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}f(t,X_t) + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t,X_t)g^2(t,X_t),$$

*which is computed according to the following Itô rules*

$$dtdt = dtdW_t = dW_tdt = 0, \quad dW_tdW_t = dt.$$

# Classical non-linear filtering theory

**System-observation model** (partial observations) in $(\Omega, \mathcal{F}, \mathbb{P})$ :

$$\boxed{\begin{aligned} dx_t &= b(x_t)dt + c(x_t)dW_t, \\ dy_t &= h(x_t)dt + dB_t, \end{aligned}}$$

- $x_0$ is $\mathcal{F}_0$-measurable r.v., $\mathcal{F}_t := \sigma\{W_s, B_s | 0 \leq s \leq t\}$
- $x_t \in \mathbb{R}$ : signal process of interest
- $y_t \in \mathbb{R}$ : observation process
- $B_t$ and $W_t$ are two independent Brownian motion
- $b$, $c$ and $h$ are bounded and Lipschitz continuous mappings

**Objective**

Describe the optimal estimation $\pi_t(f) := \mathbb{E}(f(x_t)|\mathcal{F}_t^y)$ of $f(x_t)$ in $L^2$ sense based on the observations up to time $t$, $\mathcal{F}_t^y := \sigma\{y_s : 0 \leq s \leq t\}$.

# Classical non-linear filtering theory

- **Innovations method** : show innovations process is a Wiener process, express $\pi_t(X)$ as integrals w.r.t time and innovations process (martingale techniques) ;
- **Reference probability method** : define a reference probability by Girsanov theorem, under which signals $(x_t)$ and observations $(\mathcal{F}_t^y)$ are independent.

**Theorem (Girsanov theorem)**

*Let $W_t$ be an m-dimensional $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{[0,T]}, \mathbb{P})$. Let $X_t = \int_0^t F_s ds + W_t$ for $t \in [0,T]$. Suppose that $F_t$ is Itô integrable and define*

$$\mathfrak{E}_T = exp\Big(-\int_0^T (F_s)^* dW_s - \frac{1}{2}\int_0^T \|F_s\|^2 ds\Big).$$

*If Novikov's condition $\mathbb{E}^{\mathbb{P}}\Big[exp\Big(\frac{1}{2}\int_0^T \|F_s\|^2 ds\Big)\Big] < \infty$ is satisfied, then $\{X_t\}_{t\in[0,T]}$ is an $\mathcal{F}_t$-**Brownian motion** under $\mathbb{Q}_T(A) = \mathbb{E}^{\mathbb{P}}(\mathfrak{E}_T \mathbb{1}_A)$, for all $A \in \mathcal{F}_T$.*

**Remark :** $\mathfrak{E}_T$ is a martingale under $\mathbb{P}$ (Novikov). For all $A \in \mathcal{F}_t$ with $t < T$,

$$\mathbb{Q}_T(A) = \mathbb{E}^{\mathbb{P}}(\mathfrak{E}_T \mathbb{1}_A) = \mathbb{E}^{\mathbb{P}}\big(\mathbb{E}^{\mathbb{P}}(\mathfrak{E}_T \mathbb{1}_A | \mathcal{F}_t)\big) = \mathbb{E}^{\mathbb{P}}(\mathfrak{E}_t \mathbb{1}_A) = \mathbb{Q}_t(A)$$

# Classical filtering theory : reference probability method

**System-observation model** in $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}_t := \sigma\{W_s, B_s | 0 \leq s \leq t\}$ :

$$\boxed{\begin{aligned} dx_t &= b(x_t)dt + c(x_t)dW_t, \\ dy_t &= h(x_t)dt + dB_t, \end{aligned}}$$

- $\frac{d\mathbb{Q}_t}{d\mathbb{P}} = \mathfrak{E}_t$ with $\mathfrak{E}_t = \exp\left(-\int_0^t h(x_t)dB_t - \frac{1}{2}\int_0^t h^2(x_s)ds\right)$, define $M_t = \mathfrak{E}_t^{-1}$
- $Y_t = \begin{bmatrix} W_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ \int_0^t h(x_s)ds \end{bmatrix} + \begin{bmatrix} W_t \\ B_t \end{bmatrix}$ is 2-d $\mathcal{F}_t$-B.M. under $\mathbb{Q}_t$ (Girsanov).
- $W_t$, $y_t$ and $x_0$ are **independent** ($x_0$ is $\mathcal{F}_0$-measurable r.v.).

**Kallianpur-Striebel formula (Bayes formula)**

$$\pi_t(f) := \mathbb{E}\left(f(x_t)|\mathcal{F}_t^y\right) = \frac{\mathbb{E}^{\mathbb{Q}_t}\left(M_t f(x_t)|\mathcal{F}_t^y\right)}{\mathbb{E}^{\mathbb{Q}_t}\left(M_t|\mathcal{F}_t^y\right)} =: \frac{\sigma_t(f)}{\sigma_t(1)},$$

# Zakai equation

- Itô formula :

$$M_t f(x_t) = f(x_0) + \int_0^t M_s \mathscr{L} f(x_s) ds + \int_0^t M_s \frac{df}{dx} c(x_s) ds + \int_0^t M_s f(x_s) dy_s.$$

- Taking conditional expectation :

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_t}(M_t f(x_t) | \mathcal{F}_t^y) = & \mathbb{E}^{\mathbb{Q}_t}(f(x_0) | \mathcal{F}_t^y) + \mathbb{E}^{\mathbb{Q}_t}\left( \int_0^t M_s \mathscr{L} f(x_s) ds \middle| \mathcal{F}_t^y \right) \\ & + \mathbb{E}^{\mathbb{Q}_t}\left( \int_0^t M_s \frac{df}{dx} c(x_s) dW_s \middle| \mathcal{F}_t^y \right) + \mathbb{E}^{\mathbb{Q}_t}\left( \int_0^t M_s h(x_s) f(x_s) dy_s \middle| \mathcal{F}_t^y \right).\end{aligned}$$

- Reference probability (independence) [1] :

$$\mathbb{E}^{\mathbb{Q}_t}\left( \int_0^t F_s ds \middle| \mathcal{F}_t^y \right) = \int_0^t \mathbb{E}^{\mathbb{Q}_t}(F_s | \mathcal{F}_s^y) ds,$$

$$\mathbb{E}^{\mathbb{Q}_t}\left( \int_0^t G_s dy_s \middle| \mathcal{F}_t^y \right) = \int_0^t \mathbb{E}^{\mathbb{Q}_t}(G_s | \mathcal{F}_s^y) dy_s,$$

$$\mathbb{E}^{\mathbb{Q}_t}\left( \int_0^t G_s dW_s \middle| \mathcal{F}_t^y \right) = 0.$$

1. J. Xiong, "*An introduction to stochastic filtering theory, Ch5*", Oxford, 2008

# Zakai equation

$$\mathbb{E}^{\mathbb{Q}_t}(M_t f(x_t)|\mathcal{F}_t^y) = \mathbb{E}^{\mathbb{Q}_t}(f(x_0)|\mathcal{F}_t^y) + \int_0^t \mathbb{E}^{\mathbb{Q}_t}(M_s \mathscr{L} f(x_s)|\mathcal{F}_s^y) ds + \int_0^t \mathbb{E}^{\mathbb{Q}_t}(M_s h(x_s) f(x_s)|\mathcal{F}_s^y) dy_s.$$

- For $s < t$, $\mathbb{Q}_t(A) = Q_s(A) \Rightarrow \mathbb{E}^{\mathbb{Q}_t}(M_s F(x_s)|\mathcal{F}_s^y) = \mathbb{E}^{\mathbb{Q}_s}(M_s F(x_s)|\mathcal{F}_s^y) = \sigma_s(F)$

## Zakai equation

Suppose $f \in \mathcal{C}^2$ and all derivatives of $f$ are bound,

$$\sigma_t(f) = \sigma_0(f) + \int_0^t \sigma_s(\mathscr{L} f) ds + \int_0^t \sigma_s(hf) dy_s, \quad \sigma_0(f) = \mathbb{E}^{\mathbb{P}}(f(x_0)),$$

where $(hf)(x) = h(x) f(x)$.

# Kushner-Stratonovich equation

## Kallianpur-Striebel formula

$$\pi_t(f) := \mathbb{E}\left(f(x_t)|\mathcal{F}_t^y\right) = \frac{\mathbb{E}^{\mathbb{Q}_t}\left(M_t f(x_t)|\mathcal{F}_t^y\right)}{\mathbb{E}^{\mathbb{Q}_t}\left(M_t|\mathcal{F}_t^y\right)} =: \frac{\sigma_t(f)}{\sigma_t(1)},$$

## Kushner-Stratonovich equation [1]

Suppose $f \in \mathcal{C}^2$ and all derivatives of $f$ are bound,

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(\mathscr{L} f)ds + \int_0^t \left(\pi_s(hf) - \pi_s(h)\pi_s(f)\right) d\bar{B}_s, \quad \pi_0(f) = \mathbb{E}^{\mathbb{P}}(f(x_0)),$$

where $d\bar{B}_t = dy_t - \pi_t(h)dt$ is the **innovation process**, which is a $\mathcal{F}_t^y$-Brownian motion under $\mathbb{P}$.

**Remark :** Kushner-Stratonovich equation is not a SDE for $\pi_t(f)$, since the integrants $\pi_s(\mathscr{L} f)$ and $\pi_s(hf)$ can not be expressed as functions of $\pi_s(f)$.

1. J. Xiong, "*An introduction to stochastic filtering theory, Ch5*", Oxford, 2008